The Wiener-Hopf Technique
For Penetrable Wedge Problems

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AIMS OF THIS WORK

• To illustrate the Wiener-Hopf Technique for Penetrable Wedge problems

• To present an approximate solution for the diffraction by a dielectric wedge.
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- Wiener-Hopf Formulation for Penetrable Wedge problems
- Closed Form Solutions
- Reduction of the W-H equations to Fredholm Integral Equations
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Wiener-Hopf Formulation for Dielectric Wedge problem

Normal incidence
(E-polarization)

$$E^i_z = E_o e^{jk \rho \cos(\varphi - \varphi_o)}$$

Components of the electromagnetic field:

$$E_z(\rho, \varphi) \quad H_\rho(\rho, \varphi) \quad H_\varphi(\rho, \varphi)$$
UNKNOWN IN THE WIENER-HOPF FORMULATION

Laplace transforms:

\[ V_{z+}(\eta, \varphi) = \int_0^\infty E_z(\rho, \varphi) e^{i\eta \rho} \, d\rho \quad I_{\rho+}(\eta, \varphi) = \int_0^\infty H_\rho(\rho, \varphi) e^{i\eta \rho} \, d\rho \]

\( \rho \) (radial coordinate) \( \Rightarrow \eta \) (complex frequency in the Laplace domain)

The subscript + means analytical function having regular half-plane of convergence that is an upper half-plane in the \( \eta - \text{plane} \)
The following equations have been obtained in:
Daniele, SIAM Journal of Applied Mathematics, 2003,
pp.1442-1460.
Daniele, Report ELT-2004-2 October, 2004

\[
\begin{cases}
\xi V_{z+}(\eta,0) - \omega \mu I_{\rho+}(\eta,0) = -n V_{z+}(-m,\Phi) - \omega \mu I_{\rho+}(-m,\Phi) \\
\xi V_{z+}(\eta,0) + \omega \mu I_{\rho+}(\eta,0) = -n V_{z+}(-m,-\Phi) + \omega \mu I_{\rho+}(-m,-\Phi) \\
\xi_1 V_{z+}(\eta,\pi) + \omega \mu I_{\rho+}(\eta,\pi) = -n_1 V_{z+}(-m_1,\Phi) + \omega \mu I_{\rho+}(-m_1,\Phi) \\
\xi_1 V_{z+}(\eta,\pi) - \omega \mu I_{\rho+}(\eta,\pi) = -n_1 V_{z+}(-m_1,-\Phi) - \omega \mu I_{\rho+}(-m_1,-\Phi)
\end{cases}
\]

where:

\[
\xi = \sqrt{k^2 - \eta^2} \quad \quad \xi_1 = \sqrt{\varepsilon k^2 - \eta^2}
\]

\[
m = -\eta \cos \Phi + \xi \sin \Phi \\
n = -\xi \cos \Phi - \eta \sin \Phi
\]

\[
m_1 = -\eta \cos \Phi_1 + \xi_1 \sin \Phi_1 \\
n_1 = -\xi_1 \cos \Phi_1 - \eta \sin \Phi_1
\]

\[
\Phi_1 = \pi - \Phi
\]
The previous equations define a system of generalized Wiener-Hopf equations (GWHE).

The GWHE differ from the classical W-H equations (CWHE) since they involve plus and minus analytical functions defined in different complex planes.

Also the GWHE constitute a well defined mathematical problem.

**Procedures for solving the GWHE**
Techniques for solving Generalized Wiener Hopf equation

- Generalized factorization
- Moment method
- Use of the approximate decomposition technique
- Reduction to Fredholm integral equations
Known Closed form solutions

• All the wedge problems solved (also recently) in closed form by the Sommerfeld – Malyuzhinets method, formulated in the framework of the GWHE involve a generalized factorization problem that can be solved in closed form.

• The solutions, obtained by the factorization technique and by the Sommerfeld - Malyuzhinets method, are equivalent.

• Closed form solutions do exist only for impenetrable wedge problems. Conversely also using the Generalized Wiener-Hopf no closed form solution appears to be available for dielectric wedges.

• The penetrable wedge problems amenable of exact solution involve only isorefractive wedges
Reduction of the GWHE to Fredholm Integral Equation

We obtained optimal approximate solutions for impenetrable wedge problems by reducing the GWHE to Fredholm (not singular) Integral Equation (Daniele, Report ELT-2004-2 October, 2004, Daniele-Lombardi, Days on Diffraction-2005, July 2, 2005, St.Petersburg, Russia)

In the following we will extend this technique to penetrable wedge problems.

-Note: the GWHE formulation in general has a peculiar property that provides its immediate reduction to a Fredholm equation avoiding the necessity to introduce regularization methods on singular operators.
SIMPLIFICATION OF THE GWHE

The symmetry of the problem decouples the four previous GWHE in the two system of order two. In fact the scalar factorization of the functions

\[ \xi = \xi_+ - \xi_-, n = n_- n_+, \xi_+ = \xi_- \xi_+, n_1 = n_1 n_{1+} \]

and algebraic manipulations yield:

\[
\begin{align*}
Y_{1+}(\eta) &= -\frac{n_-}{\xi_-} X_{1+}(-m) - \frac{\omega \mu}{n_+ \xi_-} X_{2+}(-m) \\
Y_{2+}(\eta) &= -\frac{n_-}{\xi_{1-}} X_{1+}(-m_1) + \frac{\omega \mu}{n_+ \xi_{1-}} X_{2+}(-m_1)
\end{align*}
\]

where:

\[
\begin{align*}
X_{1+}(-m) &= V_{z+}(-m, \Phi) + V_{z+}(-m, -\Phi) \\
X_{2+}(-m) &= I_{\rho+}(-m, \Phi) - I_{\rho+}(-m, -\Phi)
\end{align*}
\]

\[
\begin{align*}
Y_{1+}(\eta) &= 2 \frac{\xi_+}{n_+} V_{z+}(\eta, 0) \\
Y_{2+}(\eta) &= 2 \frac{\xi_{1+}}{n_{1+}} V_{z+}(\eta, \pi)
\end{align*}
\]

\[
\begin{align*}
Y_{3+}(\eta) &= n_- X_{3+}(-m) + \frac{\omega \mu}{n_+} X_{4+}(-m) \\
Y_{4+}(\eta) &= -n_- X_{3+}(-m_1) + \frac{\omega \mu}{n_{1+}} X_{4+}(-m_1)
\end{align*}
\]

\[
\begin{align*}
X_{3+}(-m) &= V_{z+}(-m, \Phi) - V_{z+}(-m, -\Phi) \\
X_{4+}(-m) &= I_{\rho+}(-m, \Phi) + I_{\rho+}(-m, -\Phi)
\end{align*}
\]

\[
\begin{align*}
Y_{3+}(\eta) &= 2 \frac{\omega \mu}{n_+} I_{\rho+}(\eta, 0) \\
Y_{4+}(\eta) &= 2 \frac{\omega \mu}{n_{1+}} I_{\rho+}(\eta, \pi)
\end{align*}
\]
SOLUTION OF FIRST SYSTEM OF GWHE:

\[
\begin{align*}
Y_{1+}(\eta) &= -\frac{n_-}{\xi_-} X_{1+}(-m) - \frac{\omega \mu}{n_+ \xi_-} X_{2+}(-m) \\
Y_{2+}(\eta) &= -\frac{n_-}{\xi_-} X_{1+}(-m_1) + \frac{\omega \mu}{n_+ \xi_-} X_{2+}(-m_1)
\end{align*}
\]

UNKNOWN:

\(Y_{1+}(\eta)\)  - Plus function in the \(\eta\) – plane

\(Y_{2+}(\eta)\)  - Plus function in the \(\eta\) – plane

\(X_{1+}(-m)\)  - Minus function in the \(m\) – plane, \(m = -\eta \cos \Phi + \sqrt{k^2 - \eta^2} \sin \Phi\)

\(X_{2+}(-m)\)  - Minus function in the \(m\) – plane, \(m = -\eta \cos \Phi + \sqrt{k^2 - \eta^2} \sin \Phi\)

\(X_{1+}(-m_1)\)  - Minus Plus function in the \(m_1\) – plane, \(m_1 = \eta \cos \Phi + \sqrt{\epsilon k^2 - \eta^2} \sin \Phi\)

\(X_{2+}(-m_1)\)  - Minus Plus function in the \(m_1\) – plane, \(m_1 = \eta \cos \Phi + \sqrt{\epsilon k^2 - \eta^2} \sin \Phi\)

**Note:** the four \(X\) functions constitute essentially two unknowns since they derive from two Laplace Transforms evaluated in the two different complex planes \(m\) and \(m_1\)
REDUCTION OF THE GENERALIZED W-H EQUATIONS TO CLASSICAL W-H EQUATIONS

Putting:

\[ \eta = -k \cos \frac{\Phi}{\pi} \arccos(-\frac{\alpha}{k}) \]

For the first equation of the system, it follows:

\[ Y_{1+}(\eta) = -\frac{n_-}{\xi_-} X_{1+}(-m) - \frac{\omega \mu}{n_+ \xi_-} X_{2+}(-m) \quad \Rightarrow \quad \overline{Y}_{1+}(\alpha) = -\frac{n_-}{\xi_-} \overline{X}_{1-}(\alpha) - \frac{\omega \mu}{n_+ \xi_-} \overline{X}_{2-}(\alpha) \]

Generalized W-H equation

Classical W-H equation

where:

\[ \overline{Y}_{1+}(\alpha) = Y_{1+}(\eta) \quad \overline{X}_{1-}(\alpha) = X_{1+}(-m) \quad \overline{X}_{2-}(\alpha) = X_{2+}(-m) \]

Note-The classical W-H equation involves plus and minus functions defined in the same \( \alpha - plane \)
REDUCTION OF THE GENERALIZED W-H EQUATIONS TO CLASSICAL W-H EQUATIONS

Putting:

\[ \eta = -k_1 \cos \frac{\Phi_1}{\pi} \arccos \left( -\frac{\beta}{k_1} \right) \]

\[ k_1 = \sqrt{\epsilon_r} \ k \]

for the second equation of the system, it follows:

\[ Y_{2+}(\eta) = -\frac{n_{1-}}{\xi_{1-}} X_{1+}(-m_1) + \frac{\omega \mu}{n_{1+} \xi_{1-}} X_{2+}(-m_1) \]

\[ \uparrow \]

\[ \ddot{Y}_{2+}(\beta) = -\frac{n_{1-}}{\xi_{1-}} \dot{X}_{1+}(\beta) + \frac{\omega \mu}{n_{1+} \xi_{1-}} \dot{X}_{2+}(\beta) \]

Generalized W-H equation

Classical W-H equation

where:

\[ \dddot{Y}_{2+}(\beta) = Y_{2+}(\eta) \quad \dot{X}_{1+}(\beta) = X_{1+}(-m_1) \quad \dot{X}_{2+}(\beta) = X_{2+}(-m_1) \]

Note-The classical W-H equation involves plus and minus functions defined in the same \( \beta \) plane
Attempts to solve in closed form the two equations:

\[
\begin{align*}
\bar{Y}_{1+} (\alpha) &= - \frac{n_-}{\xi_-} \bar{X}_{1-} (\alpha) - \frac{\omega \mu}{n_+ \xi_-} \bar{X}_{2-} (\alpha) \\
\bar{Y}_{2+} (\beta) &= - \frac{n_-}{\xi_-} \ddot{X}_{1-} (\beta) + \frac{\omega \mu}{n_+ \xi_-} \dddot{X}_{2-} (\beta)
\end{align*}
\]

require an accurate study of the mappings:

\[
\alpha (\beta) = -k \cos \left[ \frac{\pi \arccos \left( \frac{\Phi_1 \arccos \left( -\frac{\beta}{k_1} \right)}{\pi} \right)}{\Phi} \right]
\]

\[
\beta = \beta (\alpha) = -k_1 \cos \left[ \frac{\pi \arccos \left( \frac{\Phi_1 \arccos \left( -\frac{\alpha}{k} \right)}{\pi} \right)}{\Phi_1} \right]
\]

Up to now this study has not been pursued on.
PROPOSED APPROXIMATE TECHNIQUE FOR SOLVING THE PREVIOUS EQUATIONS

Taking into account the source term constituted by an incident plane wave: 
\[ E^i_z = E_o e^{jk\rho\cos(\varphi-\varphi_o)} \], a well known procedure (Vekua 1967) eliminates the plus functions and reduces the two previous classical W-H equation to the Fredholm equations:

\[ \frac{\tilde{n}_-(\alpha)}{\tilde{\xi}_-(\alpha)} \tilde{X}_1(\alpha) + \frac{\omega \mu}{\tilde{n}_+(\alpha)\tilde{\xi}_-(\alpha)} \tilde{X}_2(\alpha) + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \left[ \frac{\omega \mu}{\tilde{n}_-(\alpha)\tilde{\xi}_-(\alpha)} - \frac{\omega \mu}{\tilde{n}_+(\alpha')\tilde{\xi}_-(\alpha')} \right] \tilde{X}_2(\alpha') \frac{d\alpha'}{\alpha' - \alpha} = -\frac{\bar{T}_o}{(\alpha - \alpha_o)} \]

\[ -\frac{\tilde{n}_-(\beta)}{\tilde{\xi}_1(\beta)} \tilde{X}_1(\beta) + \frac{\omega \mu}{\tilde{n}_+(\beta)\tilde{\xi}_1(\beta)} \tilde{X}_2(\beta) + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \left[ \frac{\omega \mu}{\tilde{n}_-(\beta)\tilde{\xi}_1(\beta)} - \frac{\omega \mu}{\tilde{n}_+(\beta')\tilde{\xi}_1(\beta')} \right] \tilde{X}_2(\beta') \frac{d\beta'}{\beta' - \beta} = 0 \]

where:
\[ \bar{T}_o = j4E_o \frac{\pi}{\Phi} \quad \alpha_o = -k \cos\left(\frac{\pi}{\Phi} \varphi_o\right) \]
SOLUTION OF THE FREDHOLM EQUATIONS BY QUADRATURE

We will adopt the same scheme we have used to obtain optimal approximate solutions for the diffraction problem by arbitrary impenetrable wedges. (Daniele, Report ELT-2004-2 October, 2004). For instance we introduce the complex angular planes w and w₁:

\[ \alpha = -k \cos \frac{\pi}{\Phi} w, \quad \beta = -k_1 \cos \frac{\pi}{\Phi_1} w_1 \]

It is remarkable that in these planes we can suitably deform the contour of the integrals in order to have a good approximation of integrals with summations.
It yields the two equations having as unknowns \( P_{1,2}(hr), \ Q_{1,2}(hr) \):

\[
2 \sin \frac{\Phi}{\pi} \left( jhr - \frac{\pi}{2} \right) \frac{2}{\cosh hr} P_1(hr) + \frac{2}{\cosh hr} P_2(hr) + \frac{h}{2\pi j} \sum_{i=-A/h}^{A/h} M(hr, hi)P_2(hi) = - \frac{T_o}{k(j \sinh hr - \cos \frac{\pi}{\Phi} \phi_o)}, \quad r = 0, \pm 1, \pm 2, \ldots, \pm \frac{A}{h}
\]

\[
-2 \sin \frac{\Phi_1}{\pi} \left( jhr - \frac{\pi}{2} \right) \frac{2}{\cosh hr} Q_1(hr) + \frac{2}{\cosh hr} Q_2(hr) + \frac{h}{2\pi j} \sum_{i=-A/h}^{A/h} M(hr, hi)Q_2(hi) = 0, \quad r = 0, \pm 1, \pm 2, \ldots, \pm \frac{A}{h}
\]

where \( h \) is to be chosen as small as possible and \( A \) is to be chosen as large as possible and where:

\[
M(t,u) = If[u == t, \sec h[t] \tanh t, \left[ \frac{2}{\cosh t} - \frac{2}{\cosh u} \right] \frac{\cosh u}{\sinh t - \sinh u}]
\]
The four unknowns $P_{1,2}(hr)$, $Q_{1,2}(hr)$ are related to the two minus functions through the equations:

\[ P_1(hr) = X_{1-}(m(hr)) \]
\[ P_2(hr) = Z_o X_{2-}(m(hr)) \]
\[ Q_1(hr) = X_{1-}(m_1(hr)) \]
\[ Q_2(hr) = Z_1 X_{2-}(m_1(hr)) \]

where:

\[ m(u) = k \cos \left[ \frac{\Phi}{\pi} \left( -\frac{\pi}{2} + ju \right) + \Phi \right] \]
\[ m_1(u) = k_1 \cos \left[ \frac{\Phi_1}{\pi} \left( -\frac{\pi}{2} + ju \right) + \Phi_1 \right] \]
The figure plots $m(u)$ and $m_1(u)$ for $-\infty < u < +\infty$.

It can be shown that the minus functions $X_{1,2}(-m) = X_{1,2}-(m)$ are regular in the region delimited by the curves $\gamma$ and $\gamma_1$.

Consequently Cauchy formula applies:

$$X_{1,2}-(m_1(u)) = \frac{1}{2\pi j} \oint_{\gamma} \frac{X_{1,2}-(m)}{m - m_1(u)} \, dm = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{X_{1,2}-(m(u'))}{m(u') - m_1(u)} \frac{dm(u')}{du'} \, du'$$
The approximation of integrals with summation yields the two equations:

\[
Q_1(hr) = -\frac{h}{2\pi} \sum_{i=-A/h}^{A/h} \frac{P_1(hi) \sin[\frac{\Phi_1}{\pi}(jhi + \frac{\pi}{2}) + \Phi]}{\cos[\frac{\Phi}{\pi}(jhi + \frac{\pi}{2}) + \Phi] - \sqrt{\varepsilon_r} \cos[\frac{\Phi_1}{\pi}(jhr + \frac{\pi}{2}) + \Phi_1]}, \quad r = 0, \pm 1, \pm 2, \ldots, \pm \frac{A}{h}
\]

\[
Q_2(hr) = -\frac{h}{2\pi \sqrt{\varepsilon_r}} \sum_{i=-A/h}^{A/h} \frac{P_2(hi) \sin[\frac{\Phi_1}{\pi}(jhi + \frac{\pi}{2}) + \Phi]}{\cos[\frac{\Phi}{\pi}(jhi + \frac{\pi}{2}) + \Phi] - \sqrt{\varepsilon_r} \cos[\frac{\Phi_1}{\pi}(jhr + \frac{\pi}{2}) + \Phi_1]}, \quad r = 0, \pm 1, \pm 2, \ldots, \pm \frac{A}{h}
\]

The two new equations complete the system of order four in the four unknowns:

\[
P_{1,2}(hr), \quad Q_{1,2}(hr)
\]
\[
2 \sin \frac{\Phi}{\pi} \left( j h r - \frac{\pi}{2} \right) \frac{2}{\cosh h r} P_1(hr) + \frac{2}{\cosh h r} P_2(hr) + \frac{h}{2\pi} \sum_{i=-A/h}^{A/h} M(hr, hi) P_2(hi) = -\frac{T_o}{k (j \sinh h r - \cos \frac{\pi}{\Phi} \phi_o)},
\]

\[
2 \sin \frac{\Phi_1}{\pi} \left( j h r - \frac{\pi}{2} \right) \frac{2}{\cosh h r} Q_1(hr) + \frac{2}{\cosh h r} Q_2(hr) + \frac{h}{2\pi} \sum_{i=-A/h}^{A/h} M(hr, hi) Q_2(hi) = 0,
\]

\[
Q_1(hr) = -\frac{h}{2\pi} \sum_{i=-A/h}^{A/h} \frac{P_1(hi) \sin \left[ \frac{\Phi}{\pi} \left( j hi + \frac{\pi}{2} \right) + \Phi \right]}{\cos \left[ \frac{\Phi}{\pi} \left( j hi + \frac{\pi}{2} \right) + \Phi \right] - \sqrt{\varepsilon_r} \cos \left[ \frac{\Phi_1}{\pi} \left( j hr + \frac{\pi}{2} \right) + \Phi_1 \right]},
\]

\[
Q_2(hr) = -\frac{h}{2\pi \sqrt{\varepsilon_r}} \sum_{i=-A/h}^{A/h} \frac{P_2(hi) \sin \left[ \frac{\Phi}{\pi} \left( j hi + \frac{\pi}{2} \right) + \Phi \right]}{\cos \left[ \frac{\Phi}{\pi} \left( j hi + \frac{\pi}{2} \right) + \Phi \right] - \sqrt{\varepsilon_r} \cos \left[ \frac{\Phi_1}{\pi} \left( j hr + \frac{\pi}{2} \right) + \Phi_1 \right]},
\]

**with:**

\[r = 0, \pm 1, \pm 2, \ldots, \pm \frac{A}{h}\]

\[M(t, u) = \text{If}[u = : t, \sec h[t] \tanh t, \frac{2}{\cosh t} - \frac{2}{\cosh u} \cosh u] - \frac{2}{\cosh t - \cosh u} \cosh u]\]

\[T_o = j4E_o \frac{\pi}{\Phi}\]

\[\alpha_o = -k \cos \left( \frac{\pi}{\Phi} \phi_o \right)\]
As it happens with the similar equations used in impenetrable wedge problems, we have a good convergence of the numerical solution for typical values of $A$ and $h$ given by:

$A=10, \ h=0.25$

Similar procedure applies for the solution of the second system of slide 10.

It is important to manage the numerical solutions obtained in order to give the Sommerfeld functions in the two regions constituted by the free space and the dielectric. In the following we will limit our consideration only to the free space region:

$$-\Phi \leq \varphi \leq +\Phi$$

The Sommerfeld function in this region in terms of W-H unknowns is expressed by (Daniele, Electromagnetics, 2003, pp.223-236):

$$s_E(w) = \frac{j}{2}\left[-k \sin \omega V_{z+}(-k \cos \omega, 0) + \omega \mu I_{\rho+}(-\tau_o \cos \omega, 0)\right]$$
Up to now we have used many different complex planes.

The introduction of the Sommerfeld function makes the complex plane \( w \) the more important one. We have:

\[
\eta = -k \cos w, \quad m = k \cos(w + \Phi), \quad \alpha = -k \cos \frac{\pi}{\Phi} w
\]

Used notation: \( F_+(\eta) = F_+(-k \cos w) = \hat{F}_+(w) \)

In the dielectric region the role of \( w \) is assumed by \( w_1 \):

\[
w_1 = g(w) = \arccos \left( \frac{1}{\sqrt{\varepsilon_r}} \cos w \right), \quad w = g^{-1}(w) = \arccos \left( \sqrt{\varepsilon_r} \cos w_1 \right),
\]

We have: \( \eta = -\sqrt{\varepsilon_r} k \cos w_1 = -k \cos w \)

\[
m_1 = \sqrt{\varepsilon_r} k \cos(w_1 + \Phi_1) \quad \beta = -\sqrt{\varepsilon_r} k \cos \frac{\pi}{\Phi_1} w_1
\]
By taking into account the GWHE equations we can express the Sommerfeld function

\[ s_E(w) = \frac{j}{2} \left[ -k \sin w \hat{V}_{z+}(w, 0) + \omega \mu \hat{I}_{\rho+}(w, 0) \right] \]

in terms of the functions:

\[ \hat{X}_{1+}(w), \hat{X}_{2+}(w), \hat{X}_{3+}(w), \hat{X}_{4+}(w) \]

algebraic manipulations yield:

\[ s_E(w) = -j \frac{1}{4} \{ k \sin(w + \Phi)[\hat{X}_1(w + \Phi) + \hat{X}_3(w + \Phi)] - Z_o \hat{X}_4(w + \Phi) \} \]
THE LAST (CRITICAL) PROBLEM

The Sommerfeld function must be evaluated for every value of \( w \). It requires representations of \( \hat{X}_{1+}(w), \hat{X}_{2+}(w), \hat{X}_{3+}(w), \hat{X}_{4+}(w) \) valid everywhere.

After the evaluation of the samples \( P_{1,2}(hr), \quad Q_{1,2}(hr) \), the representations of \( \hat{X}_{1+}(w), \hat{X}_{2+}(w), \hat{X}_{3+}(w), \hat{X}_{4+}(w) \) follow from the Fredholm equations. For instance we have:

\[
\hat{X}_{1+}(w) = \frac{h}{2\pi j} \sin \frac{\pi}{\Phi} w \sum_{s=\frac{-A}{h}}^{\frac{A}{h}} M(-j(\frac{\pi}{\Phi} w - \frac{\pi}{2}), hs)P_2(hs) + \frac{\bar{T} \sin \frac{\pi}{\Phi} w_o}{k(\cos \frac{\pi}{\Phi} w - \cos \frac{\pi}{\Phi} \varphi_o)} + \\
\sqrt{\varepsilon_r} \frac{h}{2\pi j} \sin[\frac{\pi}{\Phi_1} g(w)] \sum_{s=\frac{-A}{h}}^{\frac{A}{h}} M(-j(\frac{\pi}{\Phi_1} g(w) - \frac{\pi}{2}), hs)Q_2(hs) + \\
\frac{2[\sin w + \sqrt{\varepsilon_r} \sin g(w)]}{2[\sin w + \sqrt{\varepsilon_r} \sin g(w)]} 
\]

Note- The presence of the function \( g(w) = \arccos \left( \frac{1}{\sqrt{\varepsilon_r}} \cos w \right) \), makes the Sommerfeld function a multivalued function of \( w \).
ANALYTICAL CONTINUATION

The representations $\hat{X}_{1+}(w), \hat{X}_{2+}(w), \hat{X}_{3+}(w), \hat{X}_{4+}(w)$ obtained from the Fredholm equations are valid in suitable vertical strips of the $w$-plane. An analytical continuation similar to that performed in impenetrable wedges allows to express $\hat{X}_{1+}(w), \hat{X}_{2+}(w), \hat{X}_{3+}(w), \hat{X}_{4+}(w)$ everywhere. For instance we have:

$$\hat{X}_{1+}(w) = \hat{X}_{1+}(-w), \quad \hat{X}_{2+}(w) = \hat{X}_{2+}(-w), \quad \hat{X}_{3+}(w) = \hat{X}_{3+}(-w), \quad \hat{X}_{4+}(w) = \hat{X}_{4+}(-w)$$
CONCLUSIONS

-The dielectric wedge problems can be formulated by using the Wiener Hopf technique.

- Closed form factorization techniques for solving the W-H equations seems very hard (if not impossible) to obtain for penetrable wedge problems.

-An approximate scheme of solution similar to that used for impenetrable wedges has been presented.

--This author hopes that future works could obtain an efficient W-H solution for the diffraction by a dielectric wedge.