FACTORIZATION OF MEROMORPHIC MATRICES OCCURRING IN SCATTERING PROBLEMS

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ABSTRACT

A procedure for factorizing a matrix $G(\alpha)$ whose elements are meromorphic functions of $\alpha$ is presented. Some applications are discussed.

1. INTRODUCTION

Many problems of mathematical physics lead to vector Wiener-Hopf equations that can be solved by factorizing a given square matrix of order $n$ in the form:

$$G(\alpha) = G_1(\alpha) G_2(\alpha)$$

where $G_1(\alpha)$ and $G_2(\alpha)$ and their inverses are regular and with algebraic behaviour at $\alpha = 0$ in the half-planes $\text{Im}(\alpha) > 0$ and $\text{Im}(\alpha) < 0$, respectively.

The problem of factorizing $G(\alpha)$ is very difficult to solve. In spite of the efforts of many researchers, explicit solutions have been given only in particular cases; the simpler one arises when the elements of the matrix $G(\alpha)$ are rational functions of the complex variable $\alpha = a + ja$.

These rational matrices have been studied in system theory and a complete theory of their factorization is now available. However, for what concerns the actual calculation of the factorizing matrices $G_1(\alpha)$ and $G_2(\alpha)$, it should be observed that a general procedure has been given only when the elements $G_{ij}(\alpha)$ of the matrix $G(\alpha)$ satisfy:

$$G_{ij}(\alpha) = G_{ji}(-\alpha)$$

In electromagnetic problems we are faced with a more difficult problem since the elements $G_{ij}(\alpha)$ are generally complicated multiple valued functions of $\alpha$. As before, some particular cases have been worked out, but the general problem appears to be a difficult task that many authors are hopeless that some factorizing procedure could be obtained leading to factorized matrices expressed in closed form. A little simplification of this problem arises when $G_{ij}(\alpha)$ are single value functions of $\alpha$. Such matrices are relevant to electromagnetic problems concerning closed (typically waveguide discontinuity problems) or periodic open structures.

In this paper a general procedure will be indicated enabling us to factorize $G(\alpha)$ when it is a meromorphic function of $\alpha$.

Since we are interested in electromagnetic applications we will assume only the presence of simple poles in the elements of the matrix $G(\alpha)$ and its inverse $G^{-1}(\alpha)$.

2. FACTORIZATION OF MEROMORPHIC MATRICES

For meromorphic matrices $G(\alpha)$, without loss of generality, we can assume the following representations:

$$G(\alpha) = \frac{P(\alpha)}{d(\alpha)}$$

where $P(\alpha)$ (matrix) and $d(\alpha)$ (scalar) are entire functions of $\alpha$. The form (3) allows us to express $\ln(G(\alpha))$ by the following equation:

$$\ln(G(\alpha)) = \sum_{i=0}^{n-1} \psi_i(\alpha) P(\alpha)^i$$

where $n=1$ and $\psi_i(\alpha)$ ($i=0,1,\ldots,n-1$) are scalar functions of $\alpha$ and $n$ is the order of the square matrix $G(\alpha)$. Explicit expressions of the functions $\psi_i(\alpha)$ can be obtained by different techniques, some of them will be discussed elsewhere.

The key for factorizing $G(\alpha)$ is to make an additive decompositions of the functions $\psi_i(\alpha)$:

$$\psi_i(\alpha) = \psi_i^+(\alpha) + \psi_i^-(\alpha)$$

where $\psi_i^+(\alpha)$ and $\psi_i^-(\alpha)$ are regular in the half-planes $\alpha^+ > 0$ and $\alpha^- < 0$, respectively. The additive decomposition is always possible and it is accomplished through the formulae:

$$\psi_i(\alpha) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma} \frac{\psi(u)}{u-\alpha} du$$

Taking into account the commutativity of the involved matrices, we are led to the following factorization of $G(\alpha)$:

$$G(\alpha) = \exp(\ln(G(\alpha))) = \hat{G}_1(\alpha) \hat{G}_2(\alpha)$$

where:

$$\hat{G}_1(\alpha) = \exp(\psi^+(\alpha)) I_{n-1} + \psi^+(\alpha) \psi^-(\alpha) P^0(\alpha)$$

$$\hat{G}_2(\alpha) = \exp(\psi^-(\alpha)) I_{n-1} + \psi^+(\alpha) \psi^-(\alpha) P^0(\alpha)$$

Apparently formulae (7) and (8) seem to solve the very difficult problem of factorizing a meromorphic matrix $G(\alpha)$. Actually,
even if \( \hat{G}_s(a) \) and their inverses are regular in the half-planes \( a^+0 \) and \( a^-0 \) respectively, it should be observed that they have not algebraic behaviour at \( = \). It follows we cannot use these factorized matrices to solve a vector Wiener-Hopf problem and, in order to avoid exponential behaviour at \( = \) we must introduce an entire matrix \( U(a) \) such that the new matrices are factorized with algebraic behaviour at \( = \). The explicit evaluation of the entire matrix \( U(a) \) is a formidable task. The procedure indicated in this paper circumvents this problem and apparently no better method seems available. By assuming well-posed electromagnetic problem we are dealing with field Fourier transforms vanishing at \( = \). It follows that the elements of the factorized matrices \( G(a) \) and their inverses behave as \( \hat{a}^\infty \) for \( a = (p) < 1 \). Consequently we can use the Mittag-Leffler expansion theorem and, taking into account that only simple poles are involved, we have:

\[
G_-(a) = \hat{G}_-(a)U(a) = \sum_n \hat{G}_n(a) U(a) = \sum_n \hat{G}_n(a) U(a) + \hat{G}_n(a) U(0) (11)
\]

\[
G_+(a) = \hat{G}_+(a)U(a) = \sum_n \hat{G}_n(a) U(a) + \hat{G}_n(a) U(0) (12)
\]

Where \( \hat{a}^\infty \) and \( a^\infty \) are the simple poles, with residues \( \hat{a}_n \) and \( a_n \), of the known matrices \( \hat{G}_n(a) \) and \( G_n(a) \).

From eq. (12) we have:

\[
U(a) = \sum_n \hat{G}_n(a) U(a) + \hat{G}_n(a) U(0) (13)
\]

and putting in eq. (13) \( a = \hat{a}_n \), by taking into account eq. (11) we lead to:

\[
\hat{G}_n(a) U(a) = \sum_n Q_n(a) X_n + V(a) (14)
\]

where:

\[
X_n = \hat{G}_n(a) U(a) (15)
\]

is the residue of \( \hat{G}_n(a) \) in pole \( a = \hat{a}_n \) and the matrix functions \( Q_n(a) \) and \( V(a) \) are given by:

\[
Q_n(a) = \sum_n \hat{G}_n(a) U(a) + \hat{G}_n(a) U(0) (16)
\]

\[
V(a) = \sum_n \hat{G}_n(a) U(a) + \hat{G}_n(a) U(0) (17)
\]

where \( \hat{a}^\infty \) and \( \hat{a}^\infty \) are the residues of the matrix \( \hat{G}_n(a) \) in the pole \( a = \hat{a}_n \) and the terms \( T_n \) of \( \hat{G}_n(a) \) in the pole \( a = \hat{a}_n \). Taking in to account eq. (7), eq. (14) can be rewritten in the form:

\[
\hat{G}^{-1}_n(a) U(a) = \hat{G}^{-1}_n(a) \left( \sum_n Q_n(a) X_n + V(a) \right) (18)
\]

and since the residues of both the members of eq. (18) in the simple poles \( a = a_m \) must be equal we lead to the following system of equations:

\[
X_n = R_n \left( \sum_n Q_n(a) X_n + V(a) \right) (19)
\]

where \( R_n \) is the residue of \( \hat{G}^{-1}_n(a) \) in \( a = a_m \) and \( Q_n(a_m) \) can be expressed by:

\[
Q_n(a_m) = \sum_n \hat{G}^{-1}_n(a) \left( \sum_n Q_n(a) X_n + V(a) \right) (20)
\]

Putting now:

\[
Y_n = X_n / a_n (21)
\]

we lead to the system:

\[
Y_n = R_n \left( \sum_n Q_n(a) Y_n + V(a) \right) (22)
\]

where:

\[
\sum_n \sqrt{R_n} \left( \sum_n \hat{G}^{-1}_n(a) \right) \left( \sum_n Q_n(a) Y_n + V(a) \right) (23)
\]

\[
B_n = R_n \left( \sum_n \hat{G}^{-1}_n(a) \right) \left( \sum_n Q_n(a) Y_n + V(a) \right) (24)
\]

Coefficients \( C_{nm} \) given by eq. (23) can be expressed in an alternative form which makes use of the additive decomposition of the matrix \( G(a) \):

\[
G(a) = G_r(a) + G_l(a) (25)
\]

where \( G_r(a) \) and \( G_l(a) \) are regular matrices in the half-planes \( a^+0 \) and \( a^-0 \) respectively. By taking into account the Mittag-Leffler expansion theorem we have:

\[
G_n(a) = \sum_n \hat{G}_n(a) U(a) + \hat{G}_n(a) U(0) (26)
\]

where \( \hat{a}_n \) we lead:

\[
G_n(a) - G_l(a) (27a)
\]

\[
G_n(a) = -G_r(a) (27b)
\]

Expressions (27) of \( C_{nm} \) show that the introduction of a suitable vector space, we are dealing with operators \( C^+(C_{nm}) \) which are compact; it means that even if there is an infinite number of poles \( a_j \), we can solve eq. (22) by a truncation procedure or by iterative methods. Of course a great simplification arises for rational matrices because in this case the number of the poles \( a_j \) is finite and the system (22) can be solved in closed form.

In addition expressions \( C_{nm} \) given by eq. (23) show that these coefficients depend only on the poles \( a_j \) and \( \hat{a}_n \) and the residues \( T_n \) of \( \hat{G}_n(a) \) in the poles \( a = \hat{a}_n \). It means that we do not need an expli
cit knowledge of the very complicated functions \( \psi_1(\alpha) \) to factorize the matrix \( G(\alpha) \) and the factorization procedure leading to eq. (7) has been introduced only to deduce the system (22). Of course, probably, it should be a more straightforward method to obtain this system but this author was not able to accomplish it. Expressions (4) are not however completely useless because in the representation of \( B_0 \) given by eq. (24), we must take into account the matrices \( G_0^{-1}(0) \) and \( G_0^2(0) \) that can be evaluated in the following way. Let us observe that from eq. (6) we have:

\[
(28) \quad \psi_1(0) = \psi_1(0)
\]

It follows:

\[
(29) \quad \psi_1(0) = \psi_1(0) = \psi_1(0)
\]

and consequently:

\[
(30) \quad G_0^{-1}(0) = G_0^{-1}(0) = \exp \left\{ \frac{1}{2} \left[ \psi_0(0) + \psi_0(0)^* + \cdots + \psi_0(n-1) + \psi_0(n-1)^* \right] \right\}
\]

\[
(31) \quad G_0^2(0) = G_0^2(0) = \exp \left\{ \frac{1}{2} \left[ \psi_0(0) + \psi_0(0)^* + \cdots + \psi_0(n-1) + \psi_0(n-1)^* \right] \right\}
\]

For what concerns the matrix \( U(0) \) appearing in \( B_0 \) we must observe that the eq.s (9) and (10) define an entire matrix \( B_0(0) \) a part of an arbitrary constant factor matrix; it follows then we can assume \( U(0) = 1 \).

3. APPLICATIONS

In order to give an indication of the efficiency of the previous procedure, we have successfully worked out the factorization of some rational matrices; due to the finite number of the poles \( a_n \), the system (22) has been solved in closed form. Explicit examples of these factorization will be discussed in the oral presentation of this work.

More interesting is the factorization of meromorphic matrices appearing in electromagnetic problems. In order to give a significant example let us consider the diffraction by an infinite strip grating. When strips and gaps have equal width, this problem, for plane wave excitation having an arbitrary angle of incidence, has been solved in closed form in [1]. On the contrary the case of gaps and strips having widths \( s \) and \( h \) different cannot be solved even if accurate approximate solutions have been developed in the past [2]. The Wiener-Hopf formulation of this problem leads to the factorization of the following matrix of order four:

\[
(32) \quad G_1(\alpha) = \begin{bmatrix}
\operatorname{ch} \theta & \operatorname{sh} \theta & -\operatorname{sh} \phi & \operatorname{ch} \phi \\
\operatorname{sh} \theta & \operatorname{ch} \theta & \operatorname{sh} \phi & \operatorname{ch} \phi \\
\operatorname{sh} \phi & \operatorname{ch} \phi & \operatorname{ch} \phi & \operatorname{sh} \phi \\
\operatorname{ch} \phi & \operatorname{sh} \phi & \operatorname{sh} \phi & \operatorname{ch} \phi
\end{bmatrix}
\]

where the commuting matrices \( M_s, M_h, M_d \) are given by:

\[
(33) \quad M_s(\alpha) = \exp \left( \frac{j \kappa s}{2} \alpha \right)
\]

\[
(34) \quad M_d(\alpha) = \exp \left( -\frac{j \kappa d}{2} \alpha \right)
\]

\[
(35) \quad \kappa = \sqrt{K^2 - \omega^2}
\]

\[
(36) \quad \kappa = \exp (j \kappa d \sin \phi_0)
\]

In eq. (36) the angle \( \phi_0 \) is the incidence angle.

It should be observed that the antidiagonal matrices in eq. (32) exponentially vanish as \( \alpha \to \infty \).

By factorization the matrix \( L(\alpha) = \operatorname{ch} \theta \) in the form:

\[
(37) \quad \operatorname{ch} \theta = L_-(\alpha) L_+(\alpha)
\]

eq. (32) can be rewritten:

\[
(38) \quad G_1(\alpha) = \begin{bmatrix}
L_+ & 0 \\
0 & L_-
\end{bmatrix} G(\alpha) \begin{bmatrix}
L_+ & 0 \\
0 & L_-
\end{bmatrix}
\]

where:

\[
(39) \quad G(\alpha) = \begin{bmatrix}
1 & a(\alpha) \\
b(\alpha) & 1
\end{bmatrix}
\]

\[
(40) \quad a(\alpha) = L^{-1}_-(\alpha) M_s(\alpha) M_{d\phi}^{-1}(\alpha) L^{-1}_+(\alpha)
\]

\[
(41) \quad b(\alpha) = L^{-1}_-(\alpha) M_h(\alpha) M_{d\phi}^{-1}(\alpha) L^{-1}_+(\alpha)
\]

Now we can apply the procedure indicated in the previous section to factorize \( G(\alpha) \) and, due to the exponential vanishing of \( a \) and \( b = a \alpha \), coefficients \( G_{mn} \) given by eq. (23) reduce to a sum of few terms.

The case of \( s > h \) relevant to equal width of strips and gaps allows a factorization of \( G_1(\alpha) \) in closed form because \( M_s \times M_h = M_d \). We obtain the following factorization:

\[
(42) \quad G_1(\alpha) = \begin{bmatrix}
\operatorname{ch} \phi & \operatorname{sh} \phi & -\operatorname{sh} \phi & \operatorname{ch} \phi \\
\operatorname{sh} \phi & \operatorname{ch} \phi & \operatorname{sh} \phi & \operatorname{ch} \phi \\
\operatorname{sh} \phi & \operatorname{ch} \phi & \operatorname{ch} \phi & \operatorname{sh} \phi \\
\operatorname{ch} \phi & \operatorname{sh} \phi & \operatorname{sh} \phi & \operatorname{ch} \phi
\end{bmatrix}
\]

where \( \theta_0 \) and \( \phi_0 \) are commuting matrices obtained by the additive decomposition of \( \theta \).

Both the factorizations indicated in eqs. (37) and (42) can be accomplished by the method presented in [3].

REFERENCES

