Alternative representation of dyadic Green's functions for circular cylindrical cavities with applications to the EMC characterization of space station modules

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Abstract. This paper compares two different representations of the dyadic Green's functions for circular cylindrical cavities, also valid at the source point. The representations are obtained by considering the cavity either as a circular waveguide (longitudinal representation) or as a radial waveguide (radial representation), terminated by conducting surfaces. The radial representation of dyadic Green's functions for circular cylindrical cavities is not available in the previous literature and is applied to study the field penetrating a space station module through apertures, including lateral ones, or due either to elementary sources or to wire antennas active in the module. We show that for the typical dimensions of spatial modules the radial representations give rise to more convergent series and therefore are more suitable for the computation of the cavity electromagnetic fields. The problem of obtaining the self and mutual impedances of two wire antennas located in the cavity is also discussed in general, and various results are reported for particular cases. Simplified cavity models can be effectively applied to the electromagnetic compatibility (EMC) characterization of space station modules.

1. Introduction

Accurate electromagnetic (EM) models for complex systems whose configuration varies under different operative conditions are difficult to develop. Similar difficulties are encountered when studying space station modules. These structures have the basic shape of a cylindrical tin box capped on both sides with conical frustums. Inside the module, different situations occur under different operative conditions, because the module contains various removable racks, partly or completely filled with devices or equipment which may be on or off, plus interconnecting cables with random path. In practice, these different EM configurations cannot be studied with a deterministic approach. However, if one is interested in characterizing the module from an electromagnetic compatibility (EMC) point of view, a statistic approach to this problem can be avoided by first recog-

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Figure 1. Simplified model of a space station module. To model the EM field due to the different sources active in the module, we first study the field in a circular cylindrical cavity due to electric and magnetic dipoles of moment \( P \) and \( Q \), respectively.

The longitudinal axis of the cavity of Figure 1 is the polar axis \( z \) of the cylindrical reference frame \((\rho, \phi, z)\). The cavity has length \( l \) and radius \( a \) and is filled with a linear homogeneous medium of permeability \( \mu \) and permittivity \( \varepsilon \); \( k = \omega \sqrt{\mu/\varepsilon} \) is the wavenumber, and \( Z = Y^{-1} = \sqrt{\mu/\varepsilon} \) is the impedance of the medium.

Suitable dyadic Green's functions are required to evaluate the EM fields inside a closed structure. In fact, linearity of Maxwell's equations implies that the cavity electric (E) and magnetic (H) fields in the frequency domain can be expressed in terms of the electric (J) and magnetic (M) current sources through the following integral representations:

\[
E(r) = -\int [Z(r, r') \cdot J(r')] + T_x(r, r') \cdot M(r')] \, dr',
\]

(1)

\[
H(r) = -\int [T_m(r, r') \cdot J(r')] + Y(r, r') \cdot M(r')] \, dr',
\]

(2)

where \( r \) and \( r' \) denote the observation and the source points, respectively, and \( Z, T_x, T_m, Y \) are dyadic Green's functions. The simplified model of Figure 1 allows the analytic evaluation of the Green's functions by use of well known methods, widely reported in the literature [see, e.g., Tao, 1994]. Knowledge of the Green's functions permits one to formulate integral equations to model the effect of discontinuities within the cavity. At any rate, it is of importance to separate the Green's functions into a solenoidal and a curl-free part, because the latter is highly singular when compared to the solenoidal part [Johnson et al., 1979]. The essence of the integrals resulting from this splitting is also shared by the free-space Green's functions; this has already been thoroughly discussed by Lee et al. [1980].

In spite of the fact that analytical solutions are possible one has to reconsider from a novel standpoint some well-known issues. For example, the coupling of two antennas in the cavity of Figure 1 occurs (when it does) in such a way that all the power of the transmitting antenna is taken up by the receiving one, whereas the input impedance of a single antenna in the cavity is purely reactive. Moreover, the module dimensions together with the working frequency band yield to considerable numerical difficulties. In fact, a module is typically a tin of radius \( 2.5 \) m and of \( \sim 5 \) m in length, and one is interested in a band of frequency from 3 MHz up to 1 GHz. Hence the structure at issue is multiresonating, and there is the need to develop alternative representations of the fields to choose the most rapidly convergent for the different situations of practical interest. The most frequent problems one is faced with are (1) to compute the field radiated from elementary sources, such as electric or magnetic dipoles (Figure 1); (2) to compute the field penetrating the cavity through apertures (Figures 2 and 3); and (3) to compute the field penetrating the module through a small circular aperture on the base \( z = 0 \).

Figure 2. Simplified model for the evaluation of the field penetrating the module through a small circular aperture on the base \( z = 0 \).
coupling between one transmitting and one receiving antenna, both located in the cavity (Figures 4, 5 and 6). To get accurate solutions for the different situations that might occur, we consider two alternative formulations for the dyadic Green's functions. One is based on considering the cavity as a cylindrical waveguide with short-circuit terminations at its ends (modal or longitudinal representation); the second looks at the structure as a radial waveguide (radial representation). The radial representation of the dyadic Green's functions for the geometry of Figure 1 is not available in the literature. We also show with more details than given by Daniele et al. [1998] that the radial representation is the most convenient to compute the EM field in a cavity having dimensions similar to those of a spatial module.

Section 2 reports the longitudinal representation of the dyadic Green's functions, already available in the literature though in a less general form. This representation is the starting point to derive in section 3 the radial representation. The cavity EM fields excited by elementary sources (electric and magnetic dipoles) are studied in section 4, while excitation through apertures is considered in section 5. Finally, section 6 deals with the impedance evaluation of wire antennas as well as with the coupling between

a transmitting and a receiving antenna located inside the cavity.

2. Longitudinal Representation

Since we are essentially dealing with a resonant cavity, it may look natural to represent the dyadic Green's functions in terms of cavity modes [Collin, 1966] and hence by triple summation series where each term represents one cavity mode contribution. Techniques to get the cavity mode expressions are rather well known [see, e.g., Tai, 1994; Daniele and Orefice, 1984; Morse and Feshbach, 1953]. The advantage of these representations is due to the existence of a resonant circuit model for each cavity mode, which makes the interpretation of the physical phenomena easier. However, the triple series
are slowly convergent, and this results in prohibitive computation times since, for a typical spatial module, one has thousands of modes with real resonance frequencies.

The cavity dyadic Green's functions can also be obtained by regarding the cavity as a cylindrical waveguide with short-circuit terminations in the plane $z = 0, z = l$ (Figure 1). In this approach, the Green’s functions are expressed in terms of waveguide modes by double summation series. These series converge faster than the triple series in terms of cavity modes. Procedures to build these double series representations are reported by Felsen and Marcuvitz [1973, pp. 195-200].

Equations (27) of Felsen and Marcuvitz [1973, pp. 195-200] are valid only for observation points different from the source point. To deal with $r = r'$, we slightly modified the formal procedure given by Felsen and Marcuvitz [1973, pp. 15-19] to manipulate free-space operators in time domain. In particular, the first term on the right-hand side of equation (38a) of Felsen and Marcuvitz [1973, p. 19] shows the term to be added on the right-hand side of equation (27a) of Felsen and Marcuvitz [1973, p. 195] in order to make the Green’s function expressions valid also at $r = r'$. We observed that the validity of this operative method remains also when dealing with a cylindrical cavity with perfectly conducting walls at $z = 0$ and $z = l$ and then got the expressions (4)-(9) that follow. In these equations and in the following, $\nabla$ and $\nabla_z = \nabla - \hat{z} \partial/\partial z$ are the del and the transverse del operators, respectively.

$$Z(r, r') = \frac{j k \nabla \times \hat{z}}{Z} \cdot \nabla'' r - \frac{j}{k} \hat{z} \delta(x - x')$$

$$+ \frac{j}{k} \nabla \times \hat{z} \cdot \nabla' r - \frac{j}{k} \hat{z} \delta(x - x'),$$

$$- \frac{j}{k} \nabla \times \hat{z} \cdot \nabla' r - \frac{j}{k} \hat{z} \delta(x - x'),$$

$$Y(r, r') = \frac{j k \nabla \times \hat{z}}{Y} \cdot \nabla'' r - \frac{j}{k} \hat{z} \delta(x - x')$$

$$+ \frac{j}{k} \nabla \times \hat{z} \cdot \nabla' r - \frac{j}{k} \hat{z} \delta(x - x'),$$

$$- \frac{j}{k} \nabla \times \hat{z} \cdot \nabla' r - \frac{j}{k} \hat{z} \delta(x - x'),$$

$$T_s(r, r') = \nabla \times \nabla \times \hat{z} \cdot \nabla' r - \frac{j}{k} \hat{z} \delta(x - x');$$

$$T_m(r, r') = -\nabla \times \nabla \times \hat{z} \cdot \nabla' r - \frac{j}{k} \hat{z} \delta(x - x');$$

$$L'(r, r') = \sum_R \frac{L'_r(r, r')}{(k_{1n}^2)^2}, L''(r, r') = \sum_R \frac{L''_r(r, r')}{(k_{1n}^2)^2}.$$
where, by calling $z^> = \max(z, z')$ and $z^< = \min(z, z')$, $L^i_1$ and $L^i_2$ read

$$
L^i_1(r, r') = \frac{\Phi_1(\rho)\Phi_2^*(\rho')}{c_i} \cos[kz^<\alpha] \cos[k(z^> - l)] \sin[kx^f]
$$

$$
L^i_2(r, r') = \frac{\Phi_2(\rho)\Phi_1^*(\rho')}{c_i^2} \sin[kz^<\alpha] \sin[k(z^> - l)] \sin[kx^f]
$$

(9)

In the previous expressions, $i$ stands for the double summation index corresponding to a generic mode of the circular waveguide; $k_i$, $c_i$, and the eigenfunctions $\Phi_1(\rho)$ and $\Phi_2(\rho)$ of the circular waveguide are defined by Felsen and Marcuvitz [1973, p. 264], by simply substituting in their equations the radius $a$ with $a$. $L'$ and $L''$ are related to the scalar Green's functions $G'$ and $G''$ as follows [Felsen and Marcuvitz, 1973]:

$$
G'(r, r') = -\nabla_r^2 L'(r, r'),
$$

$$
G''(r, r') = -\nabla_r^2 L''(r, r').
$$

(10)

The previous expressions are complete, converge faster than the cavity mode representations and are valid also at $r = r'$; they also take into account the $\Phi_0$ mode defined by Van Bladel [1981].

The major drawback of these longitudinal representations is that they require consideration of a large number of modes to compute the EM field. In fact, for a typical spatial module, accurate representations at 450 MHz require one to sum up more than 400 circular waveguide modes. In order to reduce the computation time it is therefore important to investigate a different representation of the dyadic Green's functions in terms of radial modes. This is the topic of section 3.

### 3. Radial Representation

The cavity of Figure 1 can also be considered as a radial waveguide with terminations at $\rho = 0$ and $\rho = a$. At least four methods are available to obtain the dyadic Green's functions by considering propagation in the $\rho$ direction: (1) One can start with the cavity mode representations and then compute in closed form the sums referring to the radial index. This procedure is similar to the one given by Tui and Rosenfeld [1976] for rectangular cavities [see also Collin, 1991, p. 383]. (2) One can start by introducing the characteristic Green's functions [Felsen and Marcuvitz, 1973, p. 273] and then deduce the alternative representations by following the method given by Felsen and Marcuvitz [1973, p. 284]. (3) One can start by considering the radial waveguide as a portion of a cylindrical surface characterized by the azimuthal and longitudinal coordinates $\phi$ and $z$, respectively. Then one can express the $E$ and $H$ fields in terms of double Fourier series in $\phi$ and $z$. This procedure has already been exploited by Basile et al. [1993] for non coincident observation, and source points and can be extended to deal also with the $r = r'$ case. (4) Finally, one can perform a symbolic inversion of the differential operators that appear in the longitudinal representations of the dyadic Green's functions given in section 2.

We have used all these four methods to prove, after very long and complex manipulations, that they lead to the same radial representations. The Appendix summarizes only the last method, which from a mathematical point of view, is not rigorous because it involves algebraic manipulations of the operators without any consideration of their domains. This method, however, turns out to be the faster to get the radial expressions that follow, where $L_i$ is used to denote the unit transverse dyadic $\rho \phi + \phi \rho$.

$$
\frac{Z(r, r')}{Z} = j\delta(\rho - \rho') \sin[kz^<] \sin[k(z^> - l)] \frac{1}{k}\frac{1}{k}
$$

$$
+ \frac{1}{k^2} \sum_{m,n=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{n}{a} \right]^2 \frac{1}{\alpha} \frac{g_{m,n}(\rho, \rho')}{\cos[m(\phi - \phi')] \sin[nz^<]} \sin[nz^<] \frac{1}{l}
$$

$$
+ \frac{1}{k^2} \sum_{m,n=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{n}{a} \right]^2 \frac{1}{\alpha} \frac{g_{m,n}(\rho, \rho')}{\cos[m(\phi - \phi')] \sin[nz^<]} \sin[nz^<] \frac{1}{l}
$$

(11)
\[ Y(r, r') = \frac{-j \delta(p - p') \cos[kz'( \cos[k(\ell - z')])]}{\sin[k\ell]} I_1 \]

\[ \frac{j}{k^2} \left( \frac{\partial^2}{\partial r^2} + k^2 \right) + \frac{j}{k} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) G''(r, r') \]

\[ \frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) G''(r, r') \right] \]

\[ J_m + \sum_{m=0}^{\infty} \frac{\cos[m(\phi - \phi')]}{I_m} \cos \left( \frac{n \pi x'}{l} \right) \]

\[ D_m(r, r') = J_m(\tau_0 \rho^*) Y_m(\tau_0) - J_m(\tau_0) Y_m(\tau_0 \rho^*) \]

\[ \tau_n = \sqrt{\rho^2 - (n \pi / l)^2} \]

\[ \varepsilon_n = \begin{cases} 1 & n = 0 \\ 2 & \text{otherwise.} \end{cases} \]

\[ \rho^c = \max(\rho, \rho') \]

\[ \rho^c = \min(\rho, \rho') \]

In the previous formulas, \( J_m \) and \( Y_m \) are the Bessel functions of order \( m \) of the first and second kind, respectively.

For cavities having the dimensions typical of a space station module, the radial representation above always allows one to accurately compute the EM fields by considering less than 40 modes at 450 MHz; this is the major advantage with respect to the use of the longitudinal representations of section 2.

4. Cavity Fields Due to Elementary EM Sources

The two elementary sources we consider are the electric and the magnetic dipoles of moment \( P = P_t \hat{z} \) and \( Q = Q_t \hat{z} \), respectively. The expression of these elementary sources located at \( r = r' \) involves a Dirac delta function

\[ J = P \delta(r - r'), \quad M = Q \delta(r - r'), \]

which, together with (1) and (2), yields

\[ E(r) = -Z(r, r') \cdot P - Y(r, r') \cdot Q, \]

\[ H(r) = -T_m(r, r') \cdot P - Y(r, r') \cdot Q. \]

Even for these simple sources the longitudinal and the radial representations of the dyadic Green's functions have different convergence rates, since the exponential decay of cutoff modes is different in the two cases. Longitudinal cutoff modes decay as \( \exp(-\kappa_l |z - z'|) \), where \( \kappa_l \) is the longitudinal propagation constant of the \( l \)th mode and \( (z - z') \) is the longitudinal-
nal distance from observation to source point. Conversely, radial cutoff modes decay as \( \exp \left\{ -\tau_n (\rho - \rho') \right\} \), where \( \rho - \rho' \) is the radial distance from observation to source point and \( \tau_n^2 = k^2 - \left( \frac{n \pi l}{l} \right)^2 \). Computation of the dipole fields by use of the longitudinal representations is discussed by Basile et al. [1995].

Similarly, after long algebraic manipulations of the radial representations, one can express the fields in terms of modal expansion coefficients \( V_{mn} \) and \( I_{mn} \) as follows:

\[
E_\phi (\rho, \phi, z) = \sum_{n=0}^{\infty} \sum_{m=\pm n} \frac{1}{2} \frac{(-i)^n}{n!} \left( \frac{n \pi l}{l} \right)^{n-1} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \frac{\partial}{\partial z} \left[ jkZ \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right] V_{mn} (\rho) \sin \left( \frac{n \pi z}{l} \right),
\]

\[
H_\phi (\rho, \phi, z) = \sum_{n=0}^{\infty} \sum_{m=\pm n} \frac{1}{2} \frac{(-i)^n}{n!} \left( \frac{n \pi l}{l} \right)^{n-1} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \frac{\partial}{\partial z} \left[ jkZ \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right] I_{mn} (\rho) \sin \left( \frac{n \pi z}{l} \right),
\]

\[
E_\nu (\rho, \phi, z) = \sum_{n=0}^{\infty} \sum_{m=\pm n} \frac{1}{2} \frac{(-i)^n}{n!} \left( \frac{n \pi l}{l} \right)^{n-1} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \frac{\partial}{\partial z} \left[ jkZ \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right] E_{\nu m} (\rho) \sin \left( \frac{n \pi z}{l} \right),
\]

\[
E_\nu (\rho, \phi, z) = \sum_{n=0}^{\infty} \sum_{m=\pm n} \frac{1}{2} \frac{(-i)^n}{n!} \left( \frac{n \pi l}{l} \right)^{n-1} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \frac{\partial}{\partial z} \left[ jkZ \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right] E_{\nu m} (\rho) \sin \left( \frac{n \pi z}{l} \right),
\]

Notice that the contribution \( E_{\nu m} \) and \( H_{\nu m} \) to the transverse field vanishes when the observation and source point have different radial coordinates, since

\[
E_{\nu m} = \frac{jkZ P_1 - \hat{z} \times Q_1}{k \sin k \hat{z}} \delta(\rho - \rho'),
\]

\[
H_{\nu m} = \frac{jkY Q_1 + \hat{z} \times P_1}{k \sin k \hat{z}} \cos \left[ \frac{k x}{2} \right] \cos \left[ \frac{k x}{2} \right] \delta(\rho - \rho').
\]

As a matter of fact, we numerically verified that for \( \rho \neq \rho' \) the radial representation converges faster than the longitudinal one. By using the superscripts \( e \) and \( m \) for the electric and the magnetic dipole source, respectively, the modal coefficients take the following expression:

\[
V_{mn} (\rho) = \frac{je_n e_m Z}{2\pi kl} \left( \cos \left( \frac{n \pi x}{l} \right) \right)^n \left( \cos \left( \frac{n \pi y}{l} \right) \right)^n \cos \left( \frac{n \pi z}{l} \right) P_1 - \frac{n \pi l}{l \tau_n} \sin \left( \frac{n \pi z}{l} \right),
\]

\[
H_{mn} (\rho) = \frac{-e_n e_m Y}{2\pi kl} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \sin \left( \frac{n \pi z}{l} \right),
\]

\[
E_{\nu m} (\rho) = \frac{je_n e_m Z}{2\pi kl} \left( \cos \left( \frac{n \pi x}{l} \right) \right)^n \left( \cos \left( \frac{n \pi y}{l} \right) \right)^n \cos \left( \frac{n \pi z}{l} \right) P_1 \cdot \nabla \left[ \left( \sin \left( \frac{n \pi z}{l} \right) \right)^n \right],
\]

\[
H_{\nu m} (\rho) = \frac{-e_n e_m Y}{2\pi kl} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \sin \left( \frac{n \pi z}{l} \right),
\]

\[
E_{\nu m} (\rho) = \frac{je_n e_m Z}{2\pi kl} \left( \cos \left( \frac{n \pi x}{l} \right) \right)^n \left( \cos \left( \frac{n \pi y}{l} \right) \right)^n \cos \left( \frac{n \pi z}{l} \right) P_1 \cdot \nabla \left[ \left( \sin \left( \frac{n \pi z}{l} \right) \right)^n \right],
\]

\[
H_{\nu m} (\rho) = \frac{-e_n e_m Y}{2\pi kl} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \sin \left( \frac{n \pi z}{l} \right),
\]

\[
E_{\nu m} (\rho) = \frac{je_n e_m Z}{2\pi kl} \left( \cos \left( \frac{n \pi x}{l} \right) \right)^n \left( \cos \left( \frac{n \pi y}{l} \right) \right)^n \cos \left( \frac{n \pi z}{l} \right) P_1 \cdot \nabla \left[ \left( \sin \left( \frac{n \pi z}{l} \right) \right)^n \right],
\]

\[
H_{\nu m} (\rho) = \frac{-e_n e_m Y}{2\pi kl} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \sin \left( \frac{n \pi z}{l} \right),
\]

\[
E_{\nu m} (\rho) = \frac{je_n e_m Z}{2\pi kl} \left( \cos \left( \frac{n \pi x}{l} \right) \right)^n \left( \cos \left( \frac{n \pi y}{l} \right) \right)^n \cos \left( \frac{n \pi z}{l} \right) P_1 \cdot \nabla \left[ \left( \sin \left( \frac{n \pi z}{l} \right) \right)^n \right],
\]

\[
H_{\nu m} (\rho) = \frac{-e_n e_m Y}{2\pi kl} \sin \left( \frac{n \pi x}{l} \right) \sin \left( \frac{n \pi y}{l} \right) \sin \left( \frac{n \pi z}{l} \right),
\]
\( V_{0h} = \frac{\varepsilon_a \varepsilon_n}{4\pi j l_h(\tau, \sigma)} \cos \left( \frac{n \pi x'}{l} \right) \hat{z} \times q, \cdot \phi' D'_{a0}(\tau, \sigma') \).  
(36)

\[
I_{hm}^{h} = \frac{j\varepsilon_a \varepsilon_n^2 \gamma}{4k l_h(\tau, \sigma)} \left[ \sin \left( \frac{n \pi x'}{l} \right) D_{ab0}(\tau, \sigma') q, \right. \\
+ \left. \frac{\pi}{l \varepsilon_n} \cos \left( \frac{n \pi x'}{l} \right) q, \cdot \phi' D_{a0}'(\tau, \sigma') \right].  
(37)
\]

As far as the transverse field components are concerned, it is possible to prove that contributions arise only from the \( m = 1 \) azimuthal mode. For sake of brevity we report only the result for the electric field due to a magnetic dipole source

\[
E_{t} = \sum_{n=1}^{\infty} \frac{1}{2\tau_n} \sin \left( \frac{n \pi x}{l} \right) \left( \sin \frac{n \pi x'}{l} \right) \\
\frac{\tau_n^2 D_{ab0}(\tau_n \sigma')}{J_0(\tau_n \sigma)} q, + \frac{\pi}{l \varepsilon_n} \cos \left( \frac{n \pi x'}{l} \right) \hat{z} \times q, \\
\left[ \left( \frac{D_{ab0}(\tau_n \sigma')}{J_0(\tau_n \sigma)} \right)' \phi' \right] \\
+ \left[ \frac{D_{ab0}(\tau_n \sigma')}{J_0(\tau_n \sigma)} \phi' \right].  
(38)
\]

The process to obtain the transverse fields at \( \rho = 0 \) is rather long. In fact one has first to prove that for \( m \neq 1 \), \( \nabla_{s} [\phi^{(m)}(\rho, \phi)] \cos[m(\phi - \phi')] = 0 \) in the limit for \( \rho = 0 \) and then get the expressions for \( \nabla_{s} [\phi^{(m)} \cos(\phi - \phi')] \) as well as for the transverse gradient of the modal coefficients \( V_{hn}^{ab} \) and \( I_{hm}^{ab} \) at \( \rho = 0 \).

5. EM Coupling Through Apertures

To show applications of the results of sections 2-4, we compute the field on the axis of the cavity when excited through apertures. Typically, apertures are present on the bases of the module (doors, Figure 2), or on its lateral surface (Figure 3), or they are holes made in the module bases for passing through wires and cables. Knowledge of the tangential electric field \( E_{o} \) on the aperture allows one to evaluate the aperture equivalent magnetic current and then, by use of the results of sections 2-4, the field in the module. The problem of how to evaluate the electric field on an aperture has been widely addressed in the literature. Depending on the dimension in wave-length of the aperture, one can either use the results provided by physical optics or by the so-called small-aperture theory. However, this theory assumes the aperture to be on a planar infinite screen, while here we want to deal with apertures on the surface of a multi-resonant cavity. To evaluate the aperture field, one needs to use a rigorous integral equation formulation not provided here. Obviously, to develop this new aperture theory, the location of the aperture on the resonant surface is of importance. In fact, for apertures located on one base of the cavity it is convenient to use the longitudinal representations to express the unknown aperture field, which is defined on the cross section of the circular waveguide. Conversely, for apertures on the cavity lateral surface, the radial representations are the most convenient since the unknown aperture field is defined on one cross section of the radial waveguide. However, one of the results of this new theory is that, even for this tin structure, physical optics yields rather good results, provided one does not work close to some resonance frequency. Therefore as a preliminary approach to study our aperture problem it is acceptable to use the classical aperture theory [Bowkamp, 1954].

5.1. Small Apertures

With reference to Figure 2, we consider a small circular aperture on the base \( z = 0 \) of the cavity. The radius of the aperture is \( c \), and the polar coordinates of the aperture center are \( (\rho', \phi') \). If the aperture is excited by a plane wave travelling in the \( z \) direction, one has [Van Bladel, 1964, p. 320, equation (10.89)]

\[
Q = Q_{e} = E_{o} \times \hat{z} = jk \frac{4}{3 \pi} \left( \frac{z}{\sqrt{z^2 - \rho_0^2}} \right) \left( \frac{z}{\sqrt{z^2 - \rho_0^2}} \right) \cdot E_{o} \times \hat{z}.  
(39)
\]

where \( E_{o} \) is the incident field, that is to say, the field one can measure by removing the cavity and \( \rho_0 \) and \( \phi_0 \) are the local circular coordinates referred to the aperture center. The electric field on the cavity axis can then be obtained by integrating on the aperture surface \( A \) the contribution coming from each elementary dipole of moment \( Q_{d} \cdot dA \); this elementary contribution is obtained by setting \( z' = 0 \) in equation (38). The integral expression one obtains can then be computed numerically. However, if \( \rho' \gg c \), on a small aperture the coordinates of the source point do not change too much and can be assumed to be equal
to the coordinates of the aperture center, $\rho'$ and $\phi'$. The electric field on the axis can be approximated as

$$E_t = \sum_{n=1}^{\infty} \frac{n \pi}{2 \lambda a} \sin \left( \frac{n \pi z}{l} \right) \int_A \mathbf{z} \times \mathbf{Q}_t \, dA$$

$$= \left\{ \begin{array}{l}
\frac{D_{l1}(\tau_n a)}{\rho' J_1(\tau_n a)} + \frac{\tau_n D'_l(\tau_n a)}{J_1(\tau_n a) \rho'} \phi' \phi'' \\
+ \frac{D_{l1}(\tau_n a)}{\rho' J_1(\tau_n a)} + \frac{\tau_n D'_l(\tau_n a)}{J_1(\tau_n a) \rho'} \phi'' \phi' \end{array} \right\}.$$ 

Conversely, with the notation of Felsen and Marcuvitz [1973, p. 263] the longitudinal representation of this field is

$$\frac{E_y}{A_E} = \frac{1}{\rho a^2} \sum_{n=1}^{\infty} \left[ \frac{\sin [\kappa_n(l-z)]}{\sin [\kappa_n(l)]} \right] J_l(\rho_n a)$$

$$+ \left( \frac{\rho_n}{\rho a} \right)^2 \frac{\sin [\kappa_n(l+z)]}{\sin [\kappa_n(l)]} J_l(\rho_n a) \right\}.$$ 

Notice how the radial representation exhibits the correct zero value of the field at $z = 0$ via the factor $\sin (n \pi z/l)$, whereas in the longitudinal representation one has the factors $\sin [\kappa_n(l-z)]$ (with $\kappa_n = \kappa_n, \kappa_n'$) which yield to the correct zero field value at $z = l$. The equivalence of the two expressions (42) and (43) by analytical means proved. It is also observed by numerical means to appropriate series truncation. As a matter of fact, for a typical spatial module, the first eight to ten terms of the radial series are enough to ensure convergence, whereas the longitudinal representation (43) requires a number of terms 8 times larger. Figure 7 reports the magnitude of the normalized field $E_y$ on the axis for the case $c = 0.1 \text{ m}$, $a = 2 \text{ m}$, $l = 4.8 \text{ m}$, $\rho_a = 1 \text{ m}$, and $E_i = 1 \text{ V/m}$, at $f = 100$, 200, and 300 MHz. These results show how the field pattern dramatically changes by changing the excitation frequency.

We proved that the radial representations can also be used to get the field penetrating the cavity through.

---

**Figure 7.** Magnitude of the normalized electric field $E_y$ on the cavity axis. The incident field ($E_i = 1 \text{ V/m}$) penetrates the cavity (with $a = 2 \text{ m}$, $l = 4.8 \text{ m}$) through a small aperture (with $c = 0.1 \text{ m}$, $\rho_a = 1 \text{ m}$). Three excitation frequencies are considered: $f = 100$, 200, and 300 MHz.
a coaxial aperture (centered on the cavity axis) on the base \( z = 0 \). Once again, even in this case, the radial representations converge more rapidly than the longitudinal ones, for given aperture fields. However, in this case, the radial representations pose some numerical difficulties at frequencies that yield \( \tau_n = 0 \).

For sake of brevity we omit to deal with coaxial apertures also because we observed no differences in the results obtained by using the longitudinal and the radial representations.

5.2. Apertures on the Lateral Surface

Space stations are built by connecting together different modules, and one possibility is indeed to join one module to the lateral surface of another module. To deal with this situation, one has to consider apertures (doors) on the lateral surface of a given module. In this case, the aperture does not lie on a planar surface, but rather it is on the lateral cylindrical surface. The field excited through the aperture is computed by integrating the aperture equivalent magnetic currents. To simplify our derivation, we will perform this integration on a planar surface having circular shape. This approximating surface can be regarded as the projection on a plane of the true aperture. A new Cartesian reference frame \( \{ X, Y, Z \} \) is attached to this plane; the unit vectors of the Cartesian frame are related to the unit vectors of the cylindrical frame \( \{ \rho, \phi, z \} \) associated with the module. With reference to Figure 3, one has \( X = \hat{x}, \ Y = \hat{\phi}', \ Z = \hat{\rho}' \), where we have indicated with \( \hat{\phi}' \) and \( \hat{\rho}' \) the value the unit vectors \( \hat{\phi} \) and \( \hat{\rho} \) have at the aperture central point \( (\rho' = a, \phi'_0, z'_0) \) on the lateral surface of the module. Hence the aperture equivalent magnetic current \( M = E_x \times (-\rho) \) might be approximated as \( M \approx \hat{Z} \times E_x \), where \( E_x \) is the electric field on the aperture surface. The field \( E_x \) can be computed as the solution of an integral equation, which will not be dealt with in this paper, or it can be obtained by use of suitable approximations. At any rate, contributions to the field in the medium arise only from the tangent components \( E_{x, X} = E_{x, z} \) and \( E_{y, Y} = E_{y, \phi}' \) of the aperture field. The field in the cavity is obtained by integral superposition of the contributions of various magnetic dipole fields \( M = Q \delta(r - r') \), where \( r' \) is a given source point on the aperture. The contribution of each of these magnetic dipoles has already been studied in section 4 and, again, for the sake of brevity, we will just study the field \( E = E_0 \delta + E_{\phi} \) on the cavity axis.

Only the azimuthal modes with \( m = 0 \) contribute to the longitudinal component of the field on the axis, and after some manipulations, by integrating the equivalent magnetic current on the aperture \( A \) one gets

\[
E_t = \frac{1}{2\pi l} \sum_{n=0}^{\infty} \frac{\tau_n}{\omega J_0(\tau_n a)} \cos \left( \frac{n\pi x'}{l} \right) \int_A \cos \left( \frac{n\pi z'}{l} \right) E_{x, X} \, dA
\]

\[
- I_1 \sin \left( \frac{n\pi x'}{l} \right),
\]

where the value of the coefficients

\[
I_1 = \int_A \cos \left( \frac{n\pi x}{l} \right) E_{x, X} \, dA,
\]

\[
I_2 = \int_A \sin \left( \frac{n\pi x}{l} \right) E_{x, X} \, dA,
\]

with \( X = z' - z_{0}' \), can be approximately evaluated by performing the integrations in the local polar coordinate system attached to the planar circle of Figure 3; this circle is obtained by projecting the true aperture on the plane tangent to the aperture center.

Only the azimuthal modes with \( m = 1 \) give non vanishing contributions to the transverse field \( E_{\phi} \) on the cavity axis; after some algebraic manipulations the elementary contribution due to a magnetic dipole of moment \( Q \) located at \( (\rho', \phi', z') \) can be expressed as

\[
E_{\phi} = \sum_{\rho} \frac{n\pi}{2\rho} \sin \left( \frac{n\pi x}{l} \right) \cos \left( \frac{n\pi z}{l} \right) \hat{Z} \times \hat{Q},
\]

\[
\left\{ \begin{array}{l}
D_{x, X} \left( J_{x, X} \frac{\tau_n}{\rho J_0(\tau_n a)} + \frac{\tau_n}{\rho J_0(\tau_n a)} \right) \hat{z}' \hat{\phi}' \\
D_{y, \phi}' \left( J_{y, \phi}' \frac{\tau_n}{\rho J_0(\tau_n a)} + \frac{\tau_n}{\rho J_0(\tau_n a)} \right) \hat{\rho}' \hat{\phi}' \\
+ \sum_{\rho} \frac{\tau_n D_{\rho, \phi}'}{2\rho J_{\phi}'(\tau_n a)} \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{n\pi z}{l} \right) Q \hat{\phi}'
\end{array} \right\}
\]

(46)
In our case, \( \rho' = a \), and one gets \( D_{x1} = D_{y1} = 0 \), \( D_{h1} = -D_{j1} = 2/(\pi a_1) \). By use of the approximations \( Q_1, \phi \approx -B_{y1} Y, (i \times Q_1, \rho \rho' \rho) \approx E_{x1} \hat{Z} \), equation (46) yields
\[
E_{x} = \sum_{n=0}^{\infty} \sum_{a} \frac{n \hat{Z}}{a} \sin \left( \frac{n \pi x}{l} \right) \left[ \frac{1}{\pi a J_{1}(\pi a)} - \frac{1}{J_{1}(\pi a)} \right] \\
\times \left[ I_{3} \cos \left( \frac{n \pi x}{l} \right) - I_{2} \sin \left( \frac{n \pi x}{l} \right) \right] \\
- \sum_{n=0}^{\infty} \frac{\hat{Y}}{\pi a J_{1}(\pi a)} \sin \left( \frac{n \pi x}{l} \right) \left[ I_{3} \cos \left( \frac{n \pi x}{l} \right) \right] \\
+ I_{4} \sin \left( \frac{n \pi x}{l} \right),
\]
(47)
where, again, the integrals
\[
I_{3} = \int_{A} \sin \left( \frac{n \pi x}{l} \right) E_{y} \, dA, \\
I_{4} = \int_{A} \cos \left( \frac{n \pi x}{l} \right) E_{y} \, dA
\]
(48)
can be computed on the projected aperture of Figure 3.

Note that the series (44) and (47) converge very rapidly because, for large value of \( a \),
\[
J_{0}(\pi a), J_{1}(\pi a), J_{1}'(\pi a) \sim \exp(\pi a/\iota) \]
whereas the integrals \( I_{1}, I_{2}, I_{3}, \) and \( I_{4} \) behave as \( 1/\iota^{\delta} \), \( \delta > 0 \).

Furthermore, if the aperture field is constant, the integral coefficients \( I_{i}, \hat{I}_{1}, \hat{J}_{1} \), and \( I_{4} \) can be written in closed form in terms of Bessel functions. The aperture field \( E_{x} \) can also be expanded in terms of the modes of a circular waveguide of radius \( c \), equal to the radius of the aperture; in this case, again, the integral coefficients have a closed form expression. By denoting with \( f_{n}^{(o)}(\rho) \), and \( f_{n}^{(e)}(\rho) \) the electric eigen-vectors of these modes (superscripts \( h \) and \( e \) are used for transverse electric (TE) and the transverse magnetic (TM) modes, respectively) one has [Pedersen and Morscetza, 1973]
\[
E_{x} = \sum_{n} \sum_{j} U_{n}^{(h)} f_{n}^{(o)}(\rho) + U_{n}^{(e)} f_{n}^{(e)}(\rho),
\]
(50)
\[
U_{n}^{(h)} = \int_{A} E_{a} \cdot f_{n}^{(e)}(\rho) \, dA.
\]
(51)

For example, by assuming that only the azimuthal modes \( i = 1 \) are present on the aperture, after lengthy manipulations one gets
\[
I_{1} = \sqrt{8 \pi} \frac{c}{\alpha} \frac{J_{1}(\alpha)}{\alpha} \sum_{i} \left\{ \frac{U_{n}^{(h)}}{\sqrt{(\chi_{1i})^{2}} - 1} \right\} \left( \frac{\alpha_{2}}{\chi_{1i}^{2} - \alpha^{2}} \right),
\]
(52)
\[
I_{2} = I_{3} = 0,
\]
(53)
\[
I_{4} = \sqrt{8 \pi} \frac{c}{\alpha} \sum_{i} \frac{U_{n}^{(h)}}{\sqrt{(\chi_{1i})^{2}} - 1} \frac{J_{1}(\alpha)}{\pi} \left( \frac{\alpha_{2}}{\chi_{1i}^{2} - \alpha^{2}} \right),
\]
(54)

6. Wire Antennas in the Cavity

In sections 6.1-6.3 we derive integral equations to model wire antennas operating inside the cavity. For simplicity we assume the wire antennas to be parallel to the z axis of the cavity. The radial representations turn out to be the most convenient to deal with these configurations.

6.1. Wire Antenna Along the Cavity Axis

The longitudinal electric field \( E_{x} \) due to a wire antenna located on the cavity axis from \( z = z_{1} \) to \( z = z_{2} \) (see Figure 4) can be expressed in terms of a line integral of its current distribution \( I(z) \) as follows:
\[
E_{x} = -\int_{z_{1}}^{z_{2}} \hat{z} \cdot \mathbf{Z}(r, r') \cdot \hat{z} I(z') \, dz',
\]
(55)
where \( \mathbf{Z}(r, r') \) is the dyadic Green's function derived in sections 2 and 3, with
\[
\hat{z} \cdot \mathbf{Z}(r, r') \cdot \hat{z} = \frac{jZ}{k} \left( \frac{\partial^{2}}{\partial z'^{2}} + k^{2} \right) G(r, r').
\]
(56)
The tangent electric field is zero on the surface of a perfectly conducting antenna, while at the feeding point \( z = z_0 \) one has \( E_z = -V_0 \delta(z - z_0) \). This leads one to consider the following integral equation:

\[
\int_{z_1}^{z_2} M(z, z')H(z')dz' = -V_0 \delta(z - z_0),
\]

(57)

whose kernel \( M(z, z') \) is obtained from (55) and (56) by assuming the source points always to be on the antenna axis \( \rho' = 0 \), whereas observation points are located on the wire surface at \( r = r_0 \bar{\rho} + z \bar{z} \), \( r_0 \) being the wire radius. By assuming \( \rho' = 0 \), only the azimuthal modes with \( m = 0 \) contribute to the radial series of \( G(\tau, \tau') \), and one gets

\[
M(z, z') = \frac{-j2}{4\pi l} \sum_{n=0}^{\infty} c_n r_0^2 \left[ \frac{J_0(\tau_0 \rho_0) Y_0(\tau_0 \alpha)}{J_0(\tau_0 \alpha)} \right] \cos \left( \frac{n\pi}{l} z \right) \cos \left( \frac{n\pi}{l} z' \right).
\]

(58)

The antenna impedance \( Z_l = V_0/H(z_0) \) can now be obtained by standard application of the method of moments with sinusoidal testing and expansion functions (Galerkin) defined on the entire wire domain. Hence the stationary expression of the impedance (one moment only) is

\[
Z_l = \frac{\int_{z_1}^{z_2} \psi_z(z)M(z, z')\psi_z(z') \, dz' \, dz}{\int_{z_1}^{z_2} \psi_z(z) \, dz} = \frac{j2}{4\pi l} \psi_z(z_0) \sum_{n=0}^{\infty} c_n r_0^2 \left[ \frac{J_0(\tau_0 \rho_0) Y_0(\tau_0 \alpha)}{J_0(\tau_0 \alpha)} \right] \cos \left( \frac{n\pi}{l} z \right),
\]

(59)

\[
f_0 = \int_{z_1}^{z_2} \psi_z(z) \cos \left( \frac{n\pi}{l} z \right) \, dz,
\]

(60)

where \( \psi_z(z) \) is the only expansion function we introduced. For example, for a quarter-wavelength antenna with feeding point \( z_0 = z_1 = 0 \), by choosing \( \psi_z(z) = \cos (2\pi z / l) \) one obtains

\[
Z_l = \frac{j2}{4\pi l} \sum_{n=0}^{\infty} c_n \left[ \frac{J_0(\tau_0 \rho_0) Y_0(\tau_0 \alpha) - J_0(\tau_0 \rho_0) Y_0(\tau_0 \alpha)}{2\tau_0^2} \right] \cos \left( \frac{n\pi}{l} \alpha \right).
\]

(61)

The previous formula is also valid for the limit in infinite \( l \) and \( n \), that is to say, for the quarter-wavelength antenna operating in free space. In these limits the series yields to a spectral integral in \( \alpha \), with \( \pi \lambda/l \to \alpha, \tau_0/l \to 2\pi, \) so that the transverse propagation wavenumber \( \tau_0 \) does not get discrete values anymore but becomes the continuous variable \( \tau \). Also, for infinite \( \alpha \), \( J_0(\tau_0 \alpha) \to j Y_0(\tau_0 \alpha) \), and (61) yields

\[
\frac{Z_l}{Z} = \int_0^{\infty} k \frac{H_2^{(2)}(\tau r_0)}{2\pi \tau^2} \cos \left( \frac{\alpha \lambda}{2\pi k} \right) \, d\alpha.
\]

(62)

For small values of \( r_0 \), the Hankel function \( H_2^{(2)}(\tau r_0) \) in the significant portion of the spectrum can be approximated by \( -jY_0(\tau r_0) \approx -2j/\tau \log (\tau r_0) \); in this case, the antenna impedance is approximated as follows:

\[
\frac{Z_l}{Z} \approx -j \frac{k}{\pi} \int_0^{\infty} \frac{\log (\tau r_0)}{\tau^2} \cos \left( \frac{\alpha \lambda}{4} \right) \, d\alpha.
\]

(63)

Numerical integration of (63) proves that in practice, the antenna impedance \( Z_l \) does not depend on the radius \( r_0 \) and is equal to \((36.67 + j21.22) \Omega\), which is the well-known result for the impedance of a quarter-wavelength antenna in free space. However, in the finite module the antenna impedance is purely reactive \((Z_l = jX_l)\). The sample numerical results reported in Table 1 were obtained with negligible computation times for a quarter-wave-

<table>
<thead>
<tr>
<th>( \lambda_0 ), m</th>
<th>( r_0 ), m</th>
<th>( X_l ), ( \Omega )</th>
</tr>
</thead>
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<tr>
<td>3</td>
<td>0.001</td>
<td>-22.38</td>
</tr>
<tr>
<td>3</td>
<td>0.005</td>
<td>-13.10</td>
</tr>
<tr>
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<td>0.010</td>
<td>-9.28</td>
</tr>
<tr>
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<td>0.010</td>
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<td>25.79</td>
</tr>
<tr>
<td>1/2</td>
<td>0.005</td>
<td>24.45</td>
</tr>
<tr>
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</tr>
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<tr>
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<td>123.81</td>
</tr>
<tr>
<td>1/3</td>
<td>0.010</td>
<td>121.00</td>
</tr>
</tbody>
</table>
length monopole with \( z_1 = z_0 = 0 \) and \( z_2 = \lambda/4 \), mounted in a cavity of length \( l = 4.8 \) m and radius \( a = 2 \) m.

6.2. Off-axis Wire Antenna

If the axis of the wire antenna is at \( \rho = \rho_2 >> r_0, \phi = \phi_1 \) (see Figure 5), the kernel \( M(z, z') \) of the integral equation (57) becomes

\[
M(z, z') = \frac{-jZ}{2\pi kl} \sum \sum \sum \sum \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4 (r_1 - r_0, r_1 - r_0) \]
\[
\cdot \cos \left( \frac{n \pi \rho}{l} \right) \cos \left( \frac{n \pi \phi}{l} \right).
\] (64)

In this case, the stationary value of the antenna impedance is

\[
Z_i = \frac{jZ}{4\pi k l (\omega)} \sum \sum \sum \sum \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4 (r_1 - r_0) \]
\[
\cdot \cos \frac{n \pi \rho}{l} \cos \frac{n \pi \phi}{l}.
\] (65)

with \( f_0 \) defined as in (60). For a quarter-wavelength antenna with feeding point at \( z_2 = z_0 = 0 \), one obtains

\[
Z_i = \frac{4\pi k l (\omega)}{f_0} \sum \sum \sum \sum \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4 (r_1 - r_0) \]
\[
\cdot \cos \left( \frac{n \pi \rho}{l} \right) \cos \left( \frac{n \pi \phi}{l} \right).
\] (66)

6.3. Coupling of Two Wire Antennas Parallel to the Cavity Axis

With reference to Figure 6 we study the coupling between two wire antennas: the first is located on the module axis with point at \( z = z_0 \), while the axis of the second antenna is \( \rho_2 \), and its feeding point is at \( z = z_2 \). The wire radii are \( r_0 \) and \( r_2 \), respectively. For this configuration, use of (55) and (56) yields the coupled integral equations

\[
\begin{align*}
V_1 (z) &= \int_{z_0}^{z_1} M_{11}(z, z') I_1(z') \, dz' + \int_{z_2}^{z_0} M_{12}(z, z') I_2(z') \, dz', \\
V_2 (z) &= \int_{z_0}^{z_1} M_{21}(z, z') I_1(z') \, dz' + \int_{z_2}^{z_0} M_{22}(z, z') I_2(z') \, dz'.
\end{align*}
\] (67)

\[
\begin{align*}
V_1 (z) &= \int_{z_0}^{z_1} M_{11}(z, z') I_1(z') \, dz' + \int_{z_2}^{z_0} M_{12}(z, z') I_2(z') \, dz', \\
V_2 (z) &= \int_{z_0}^{z_1} M_{21}(z, z') I_1(z') \, dz' + \int_{z_2}^{z_0} M_{22}(z, z') I_2(z') \, dz'.
\end{align*}
\] (68)

\[
\begin{align*}
M_{11}(z, z') &= \frac{-jZ}{4\pi k l (\omega)} \sum \sum \sum \sum \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4 (r_1 - r_0, r_1 - r_0) \]
\[
\cdot \cos \frac{n \pi \rho}{l} \cos \frac{n \pi \phi}{l},
\] (69)

\[
\begin{align*}
M_{12}(z, z') &= \frac{-jZ}{4\pi k l (\omega)} \sum \sum \sum \sum \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4 (r_1 - r_0, r_1 - r_0) \]
\[
\cdot \cos \frac{n \pi \rho}{l} \cos \frac{n \pi \phi}{l},
\] (70)

\[
\begin{align*}
M_{22}(z, z') &= \frac{-jZ}{4\pi k l (\omega)} \sum \sum \sum \sum \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4 (r_1 - r_0, r_1 - r_0) \]
\[
\cdot \cos \frac{n \pi \rho}{l} \cos \frac{n \pi \phi}{l},
\] (71)

\[
\begin{align*}
M_{21}(z, z') &= \frac{-jZ}{4\pi k l (\omega)} \sum \sum \sum \sum \sin \phi_1 \sin \phi_2 \sin \phi_3 \sin \phi_4 (r_1 - r_0, r_1 - r_0) \]
\[
\cdot \cos \frac{n \pi \rho}{l} \cos \frac{n \pi \phi}{l},
\] (72)

Application of the moments method yields the following simple circuit representation in terms of a 2 x 2 impedance matrix:

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\] (73)

where \( I_1,2 \) are the currents at the feeding points. Approximate values for the coefficients of the impedance matrix can be obtained by using only two expansion functions (Galerkin): \( \phi_1(z) \) for \( I_1(z) \) and \( \phi_2(z) \) for \( I_2(z) \). The stationary values are
Z_{11} = -\frac{jZ}{4k\psi(r_0)} \sum_{n=0}^{\infty} \frac{e_{n} r_0^2 D_{m}(r_0)}{J_{n}(r_0)} \int_{1}^{1} \psi_{n}(r_{1}) \psi_{n}(r_{2}) \frac{r_{1}^{2}}{J_{n}(r_{0})} dr_{1} \quad (74)

Z_{m} = -\frac{jZ}{4k\psi'(r_0)} \sum_{n=0}^{\infty} \frac{e_{n} r_0^2 D_{m}(r_0)}{J_{n}(r_0)} \int_{1}^{1} \psi_{n}(r_{1}) \psi_{n}(r_{2}) \frac{r_{1}^{2}}{J_{n}(r_{0})} dr_{1} \quad (75)

Z_{22} = \frac{jZ}{4k\psi'(r_0)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{e_{n} e_{m} r_0^2 D_{m}(r_0)}{J_{n}(r_0)} \int_{1}^{1} \psi_{m}(r) \psi_{n}(r) \frac{r_{1}^{2}}{J_{n}(r_{0})} dr_{1} \quad (76)

f_{m} = \int_{1}^{1} \psi_{m}(r) \cos \left( \frac{\pi r}{T} \right) dr_{1} \quad (77)

f_{2n} = \int_{1}^{1} \psi_{2n}(r) \cos \left( \frac{\pi r}{T} \right) dr_{1} \quad (78)

Let us consider, for example, two quarter-wavelength antennas with feeding points on the module bases z_{11} = 0, z_{22} = l \text{ and } z_{11} = 0, z_{22} = \lambda/4, z_{22} = (l - \lambda/4). In this case, upon setting \psi_{1}(z) = \cos(2\pi z/\lambda) and \psi_{2}(z) = \cos(2\pi(l - z)/\lambda), one gets

Z_{11} = \frac{j\pi Z}{2k} \sum_{n=0}^{\infty} \frac{e_{n} D_{m}(r_0)}{r_{0}^{2} J_{n}(r_0)} \cos\left( \frac{\pi n \lambda}{4l} \right) \quad (79)

Z_{m} = \frac{j\pi Z}{2k} \sum_{n=0}^{\infty} \frac{e_{n} e_{m} D_{m}(r_0)}{r_{0}^{2} J_{n}(r_0)} \cos\left( \frac{\pi n \lambda}{4l} \right) \quad (80)

Z_{22} = \frac{j\pi Z}{2k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{e_{n} e_{m} D_{m}(r_0)}{r_{0}^{2} J_{n}(r_0)} \cos\left( \frac{\pi n \lambda}{4l} \right) \quad (81)

It is of importance to notice that the impedance matrix in (73) is, in general, purely imaginary. Therefore, when the second antenna is loaded with an impedance Z_{2}, the active power on Z_{2} is equal to the outgoing active power of the first antenna. If the circuit model of the signal generator of the first antenna is available and the load Z_{2} is known, the power balance for the transmitting system can be obtained very easily with circuit consideration, provided the impedance coefficients Z_{11}, Z_{m}, and Z_{22} have been computed.

7. Conclusion and Future Work

This paper compares two representations of the dyadic Green's functions of a circular cylindrical cavity which are used to study EMC problems in space station modules. Different applications of the derived radial representations are considered. In particular, we study the field penetrating the cavity through apertures or due to elementary sources on or to wave antennas. The problem of obtaining the self- and mutual impedances of two wire antennas placed in the cavity is also discussed in general, and particular results are reported.

Future work will consider in more detail the problem of apertures on the cavity surface, to model the aperture field in terms of integral equations. In fact, physical optics is unable to provide good results, especially if the frequency of the exciting signal happens to be close to a cavity resonance frequency.

Appendix

A procedure to obtain the radial representations (11)-(14) of the dyadic Green's functions Z, Y, T_{r}, and T_{m} is based on the completeness relationships [Fein and Marcuvitz, 1973, p. 244, equation (16); p. 245, equation (23); p. 256, equation (50b)]

\[ \delta(z-z') = \frac{1}{\lambda} \sum_{n=0}^{\infty} e_{n} \sin \left( \frac{n \pi z}{\lambda} \right) \sin \left( \frac{n \pi z'}{\lambda} \right) \]

\[ \delta(\phi-\phi') = \frac{1}{2\pi} \sum_{n=0}^{\infty} e_{m} \cos \left( \frac{n \pi \phi}{\lambda} \right) \cos \left( \frac{n \pi \phi'}{\lambda} \right) \]

where, as already stated in (20), e_{n} = 1 if n = 0, and e_{n} = 2 otherwise. The procedure starts by deriving the radial expressions (15) and (16) of the scalar Green's functions G = G', G''. These functions satisfy the equation \( \nabla^2 G + k^2 G = -\delta(r-r') = -\delta(\phi-\phi') \delta(z-z') \delta(\rho-\rho') \) with boundary conditions (3). Formal inversion of the differential equation for G' together with the completeness relationships (A1) yields
\[ G'(r, r') = -\frac{1}{2\pi l} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ c_{n+m} \frac{1}{\sqrt{n^2 + k^2}} \frac{\delta(\rho - \rho')}{\rho} \right] \cdot \cos[m(\phi - \phi')] \cos \left( \frac{n\pi z}{l} \right) \cos \left( \frac{n\pi z'}{l} \right) \right], \quad (A2) \]

where, owing to the \( z \)-dependence of (A2), the operator \( \nabla^2 + k^2 \) can be simplified as

\[ \nabla^2 + k^2 = \nabla_1^2 + \frac{\partial^2}{dz^2} + k^2 = \nabla_1^2 + \tau_n^2, \quad (A3) \]

\[ \tau_1^2 = k^2 - \left( \frac{n\pi}{l} \right)^2. \quad (A4) \]

To further develop (A2), we now define \( g_{mn}(\rho, \rho') \) as the radial function that satisfies

\[ -\frac{1}{\nabla_1^2 + \tau_n^2} \frac{\delta(\rho - \rho')}{\rho} = -g_{mn}(\rho, \rho'), \quad (A5) \]

with boundary conditions \( g_{nn}(\rho, \rho') = 0 \) at \( \rho = a \) (see equation (3)). The expression of \( g_{mn}(\rho, \rho') \) reported in (17) and (18) is obtained by applying the relevant technique described by Felsen and Marcuvitz [1973, p. 278] to

\[ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} + \tau_n^2 \right] g_{mn}(\rho, \rho') = -\frac{\delta(\rho - \rho')}{\rho}. \quad (A6) \]

The radial representation of \( G' \) is obtained by using (A3) together with the expression of \( g_{mn}(\rho, \rho') \) to rearrange (A2), as reported in (15). The radial expression (16) is similarly obtained by applying this procedure to the differential equation for \( G'' \).

In order to derive (11)-(14) from the longitudinal representations (4)-(7) of the dyadic Green's functions \( Z, Y, T, \) and \( T_m \) we have now to provide the radial representations of the two functions \( L' \) and \( L'' \) defined in (10). It is readily verified that these functions read

\[ L'(r, r') = \frac{1}{2\pi l} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ c_n \varepsilon_n \varepsilon_m T_m(\rho, \rho') \right] \cdot \cos[m(\phi - \phi')] \cos \left( \frac{n\pi z}{l} \right) \cos \left( \frac{n\pi z'}{l} \right), \quad (A7) \]

\[ L''(r, r') = \frac{1}{2\pi l} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{\varepsilon_n \varepsilon_m}{\sin \left( \frac{n\pi z}{l} \right) \sin \left( \frac{n\pi z'}{l} \right)} \right] \cdot \cos[m(\phi - \phi')] \cos \left( \frac{n\pi z}{l} \right) \cos \left( \frac{n\pi z'}{l} \right), \quad (A8) \]

where, according to (10), \( \lambda_{mn} = \lambda_m, \mu_{mn} = \mu_m \) has to satisfy the constraint (with \( \gamma_{nn} = \gamma_m, \beta_{nn} = \beta_m \))

\[ \lambda_{mn}(\rho, \rho') = \frac{1}{\nabla_1^2} \gamma_{mn}(\rho, \rho') \]

\[ = \frac{1}{\nabla_1^2} \left[ \frac{1}{\tau_n^2} \frac{1}{\tau_n^2} \delta(\rho - \rho'). \quad (A9) \right. \]

The boundary conditions and the formal identity

\[ \frac{1}{\nabla_1^2} \frac{1}{\nabla_1^2 + \tau_n^2} = \frac{1}{\tau_n^2} \frac{1}{\tau_n^2} \left( \nabla_1^2 + \tau_n^2 \right) \quad (A10) \]

yield

\[ \lambda_{mn}(\rho, \rho') = \frac{1}{\tau_n^2} \nabla_1^2 \delta(\rho - \rho') + \frac{1}{\tau_n^2} \gamma_{mn}(\rho, \rho'), \quad (A11) \]

\[ \lambda_{mn}(\rho, \rho') = \frac{1}{\tau_n^2} \nabla_1^2 \delta(\rho - \rho') + \frac{1}{\tau_n^2} \gamma_{mn}(\rho, \rho'). \quad (A12) \]

By substituting (A10)-(A12) into (A7) and (A8) and by exploiting the identity

\[ \frac{\varepsilon_n}{\sin \left( \frac{n\pi z}{l} \right) \sin \left( \frac{n\pi z'}{l} \right)} \sum_{n=0}^{\infty} \varepsilon_n \cos \left( \frac{n\pi z}{l} \right) \cos \left( \frac{n\pi z'}{l} \right) \]

\[ = \frac{1}{\tau_n^2 \sin \left( \frac{n\pi z}{l} \right) \sin \left( \frac{n\pi z'}{l} \right)} \sum_{n=0}^{\infty} \varepsilon_n \cos \left( \frac{n\pi z}{l} \right) \cos \left( \frac{n\pi z'}{l} \right) \]

\[ = \frac{1}{\tau_n^2 \sin \left( \frac{n\pi z}{l} \right) \sin \left( \frac{n\pi z'}{l} \right)} \sum_{n=0}^{\infty} \varepsilon_n \cos \left( \frac{n\pi z}{l} \right) \cos \left( \frac{n\pi z'}{l} \right) \]

one gets
\[ L'(r, r') = \frac{1}{2\pi l} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon_n e_m}{r_n^2} \cos \left( \frac{\pi n}{l} z \right) \cos \left( \frac{\pi r}{l} \right) \delta(r - r') \]

\[ + \frac{1}{\nabla_t^2} \delta(r - r') \cos \left( k z^c \right) \frac{\cos \left( k z^c \right) \cos \left( k z^{c'} \right) - k z^{c'} \cdot \delta(r - r')}{k \sin(k)} \]

\[ L''(r, r') = \frac{1}{2\pi l} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon_n e_m}{r_n^2} \cos \left( \frac{\pi n}{l} z \right) \cos \left( \frac{\pi r}{l} \right) \delta(r - r') \]

\[ + \frac{1}{\nabla_t^2} \delta(r - r') \cos \left( k z^c \right) \frac{\cos \left( k z^c \right) \cos \left( k z^{c'} \right) - k z^{c'} \cdot \delta(r - r')}{k \sin(k)} \]

(A14)

So far we have derived the radial expressions of the Green's functions \( G \) and \( L \). The last item required to derive the dyadic Green's functions (see equations (4) and (5)) is provided by the identities

\[ \nabla_t \nabla_t' \sum_i \frac{\Phi_i(\rho) \Phi_i'(\rho')}{(k_i l)^2} \delta(z - z') \]

\[ = -\nabla_t \nabla_t' \delta(r - r'), \]

\[ \nabla_t \nabla_t' \sum_i \frac{\Phi_i(\rho) \Phi_i'(\rho')}{(k_i l)^2} \delta(z - z') \]

\[ = -\nabla_t \nabla_t' \delta(r - r'). \] (A16)

The first step consists in proving

\[ \nabla_t \nabla_t' \nabla' \nabla' \nabla_t' \delta(r - r') \]

\[ = \left[ \left( \nabla_t \frac{\partial}{\partial z} - \hat{z} \nabla_t^2 \right) \left( \nabla_t \frac{\partial}{\partial z'} - \hat{z} \nabla_t^2 \right) \right] L'(r, r') \]

\[ = \nabla_t \nabla_t' \frac{\partial}{\partial z} \frac{\partial}{\partial z'} L'(r, r') \]

\[ + \nabla_z \frac{\partial}{\partial z} \frac{\partial}{\partial z'} \nabla_t \nabla_t' L'(r, r') \] (A17)

by substituting \( \nabla_t^2 L' = -L'' \) (see equation (10)). Then, by recalling that \( (n \pi/l)^2 = k^2 - \tau_n^2 \) (see equation (A4)), use of (A13) and (A14) yields

\[ \nabla_t \nabla_t' \frac{\partial}{\partial z} \frac{\partial}{\partial z'} L'(r, r') = k \nabla_t \nabla_t' \frac{\pi n}{l} \sin(k z^c) \frac{\cos(k z^{c'}) - k z^{c'} \cdot \delta(r - r')}{k \sin(k)} \]

\[ \sin(k z^c) \sin(k z^{c'}) - \nabla_t \nabla_t' \frac{\pi n}{l} \sin(k z^c) \frac{\cos(k z^{c'}) - k z^{c'} \cdot \delta(r - r')}{k \sin(k)} \]

\[ + \nabla_t \nabla_t' \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \varepsilon_n e_m \left( \frac{2 \pi r}{l} \frac{\pi n}{l} \right)^2 g_{mn}(\rho, \rho') \]

\[ \cdot \cos(m \phi - m' \phi') \sin \left( \frac{\pi n \tau_n}{l} \right) \sin \left( \frac{\pi n \tau_n'}{l} \right). \] (A15)

Finally, by noticing that

\[ \left[ \nabla_t \nabla_t' + \frac{\nabla_t \times \hat{z} \nabla_t' \times \hat{z} \nabla_t'}{\nabla_t^2} \right] \delta(\rho - \rho') = -L_t \delta(\rho - \rho'), \] (A19)

(A15) and (A18) yield

\[ \frac{j}{k} \left[ \nabla_t \nabla_t' \delta(r - r') + \nabla_t \nabla_t' \frac{\partial}{\partial z} \frac{\partial}{\partial z'} L'(r, r') \right] + \]

\[ jk \nabla \times \hat{z} \nabla' \times \hat{z} L''(r, r') \]

\[ = j \delta(\rho - \rho') \sin(k z^c) \frac{\cos(k z^{c'}) - k z^{c'} \cdot \delta(r - r')}{k \sin(k)} \]

\[ + \frac{j}{k} \nabla_t \nabla_t' \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \varepsilon_n e_m \left( \frac{\pi n \tau_n}{l} \right)^2 g_{mn}(\rho, \rho') \]

The radial representation (11)-(14) of the dyadic Green's functions \( Z \), \( Y \), \( T_\tau \), and \( T_m \) can be obtained by substituting (A14) and (A15) into (4)-(7) and by use of (A16). Some details and the remaining intermediate steps required to get the radial expressions are shown by explicitly deriving from (4) the dyadic function \( Z \) reported in (11); similar application of the same procedure to (5)-(7) leads to (12)-(14).
\[
\begin{align*}
&\cdot \cos(m(\phi - \phi')) \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{n'\pi x'}{l} \right) \\
&+ jk \frac{\nabla x \times \nabla x \times \nabla}{2\iota} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varepsilon_{n,m}}{r_0^3} \delta_m(\rho, \rho') \\
&\cdot \cos(m(\phi - \phi')) \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{n'\pi x'}{l} \right) \right). \quad (A20)
\end{align*}
\]

Now, by substituting (A16) and (A17) into (4) one has

\[
\frac{Z(r', r')}{Z} = \frac{1}{k} \left[ \nabla \cdot \nabla \gamma + \nabla \cdot \nabla \gamma \times \nabla \Gamma \times \nabla \Gamma \right] + jk \nabla \times \nabla \gamma \times \nabla \Gamma \times \nabla \Gamma
\]

\[
+ \frac{j}{k} \left[ \nabla \gamma \cdot \nabla \gamma + \nabla \gamma \cdot \nabla \gamma \right] \Gamma(r, r')
\]

\[
- \frac{j}{k} \left[ \delta(r - r') + \nabla \gamma \Gamma(r, r') \right]. \quad (A21)
\]

where, for the last term on the right-hand side of

\[
(A21), \quad \text{one has}
\]

\[
\left[ \delta(r - r') + \nabla \gamma \Gamma(r, r') \right] = - \left( \frac{\partial^2}{\partial x^2} + k^2 \right) \Gamma(r, r').
\]

The radial expression (11) of the dyadic function

\[
Z(r, r') \text{ is readily obtained by direct substitution of}
\]

(A20) and (A22) into (A21).

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