Radiation by arbitrary sources in anisotropic stratified media

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The radiation of an arbitrary source in a stratified system including anisotropic slabs is considered. Starting from an abstract-operator formulation given by Bresler and Marcuvitz [1956, 1957], the transverse field is evaluated by using a Fourier-integral representation. The Green's function is expressed through the evaluation of the source-free solutions. The scattered field and the plane-wave response are examined. An example is given concerning the field radiated in the presence of an anisotropic fertile slab. The far field is evaluated through a saddle-point integration.

INTRODUCTION

Electromagnetic problems require the evaluation of the electromagnetic field produced by prescribed sources in a given medium. The complexity of the problem depends essentially on the properties of the medium in which the field propagates. Practical problems are usually defined in nonhomogeneous regions. The medium may change its properties continuously (troposphere, ionosphere, particular lenses), or it may present surfaces of discontinuity (stratified media, guided waves, systems, and scatterers). In both cases, the problems can be stated in terms of electromagnetic fields defined in the whole space and obeying the same system of Maxwell equations with coefficient functions of the point.

A first step in approaching these problems is to consider media stratified perpendicularly to a fixed direction, with properties changing continuously or discontinuously. A fundamental contribution has been given by Marcuvitz and Schwinger [1951] by stating the problem in terms of the transverse field components, which are continuous on discontinuity sections and can be expanded in terms of z-independent transverse-vector eigenfunctions and of scalar functions of z. Such scalar functions obey transmission-line equations.

The Marcuvitz and Schwinger procedure can be extended to anisotropic media in some particular cases [Arbel and Felsen, 1963], but in many other situations it may fail. A more general approach must be used, like the abstract-operator formulation developed by Bresler and Marcuvitz [1956, 1957]. In this formulation, the Maxwell equations appear as a single operator equation. Through an abstract procedure the transverse-field equations are obtained.

This paper deals with the evaluation of the field excited by arbitrary sources in open waveguide regions, that is, in systems which present an unbounded cross section. The medium is stratified along the z axis and may be anisotropic somewhere. Starting from the transverse-field equations of Bresler and Marcuvitz, a 2-transmission representation is obtained. This has been done by using a two-dimensional integral representation, which is equivalent to introducing a Fourier basis in the two-Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y$, a factor of the product space where the transverse equations are defined. The transformed wave vector obeys an inhomogeneous first-order differential equation, the properties of which have been widely investigated [Paine, 1965]. The appropriate Green's function has been evaluated through the matricant and through source-free solutions. Finally, the field scattered by stratified media has been considered, and the plane-wave response, obtained. An example is given in which a far-field evaluation has been obtained with a saddle-point integration.

The matrix formulation used in the present paper overcomes the difficulties arising when the conventional transmission-line concept fails. Moreover, the problem is stated in a language which can be handled by a computer in a straightforward way.

The rationalized mks system of units has been adopted, and the time-dependence factor $\exp (-iat)$ has been omitted. The following notations will be used:
\[ \omega = \frac{2\pi f}{c}, \text{ angular frequency} \]
\[ \epsilon_0, \mu_0, \text{ electric permittivity and magnetic permeability in a vacuum} \]
\[ k_0 = (\mu_0 / \epsilon_0)^{1/2}, \text{ wave number in a vacuum} \]
\[ Z_0 = (\mu_0 / \epsilon_0)^{1/2}, \text{ intrinsic impedance of free space} \]
\[ x, y, z, \text{ Cartesian coordinates} \]
\[ \hat{z}, \hat{y}, \hat{x}, \text{ the associated unit vectors} \]
\[ x = x \hat{x} + y \hat{y} + z \hat{z}, \text{ polar vector in space} \]
\[ e = e \hat{x} + y \hat{y}, \text{ polar vector in transverse planes} \]
\[ l = \hat{x} + \hat{y} + \hat{z} \]
\[ i = \hat{x} + \hat{y} \]
\[ \nabla_1 = \nabla - \frac{e}{(\partial / \partial z)} \text{ transverse gradient operator} \]
\[ E, H, J, \text{ electric and magnetic field vectors;} \]
\[ E_i = E, H_i = H \text{ transverse electric and magnetic vectors} \]
\[ \begin{align*}
I & \rightarrow \\
R & \rightarrow
\end{align*}
\[
\begin{bmatrix}
& -1 - (1/a\mu) \hat{x} \times (1/a\mu) \hat{z} \times \\
(1/a\mu) \hat{y} \times (1/a\mu) \hat{z} \times \\
(1/a\mu) \hat{y} \times (1/a\mu) \hat{z} \times
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 - (1/a\epsilon) \hat{x} \times (1/a\mu) \hat{y} \times \\
(1/a\epsilon) \hat{x} \times (1/a\mu) \hat{y} \times \\
(1/a\epsilon) \hat{x} \times (1/a\mu) \hat{y} \times
\end{bmatrix}
\]
\[ x, y \text{ relative electric permittivity and magnetic permeability dyadics, in addition} \]
\[ x_i = 1, x^e, x^s, x_t = x \hat{x} \hat{z} \hat{z}, x_s = x \hat{x} \hat{x} \hat{x}, \text{ and similarly for } y \]
\[ \text{the wave vector } \psi', \text{ the current vector } \psi'', \text{ and the operators } \Theta \text{ and } \Gamma', \text{ are represented by:} \]
\[ \psi' \rightarrow E \]
\[ \psi'' \rightarrow Z J \]
\[ \Theta \rightarrow \\
\[ \Gamma' \rightarrow \]
\[ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ \text{primed and unprimed letters (e.g. } a' \text{ and } a) \text{ indicate the same abstract element when the natural basis} \]
\[ \text{or the Fourier basis is assumed in } 3C = 3C_e \otimes 3C_s. \]
\[ \text{In our notation such quantities are related through} \]
\[ n(\delta; z) = \int n'(\delta; g(z) \times z) (1 - i k_0 d g) d g \]
\[ n'(\delta; z) = \int n'(\delta; z) \exp (i k_0 d g) d g \]
\[ d = x \hat{x} + y \hat{y} + z \hat{z} \]
\[ \mathcal{A}(A) \text{ and } \mathfrak{d}(A) \text{ are the domain and the range of the operator } A; \]
\[ A^* \text{ is the transpose of } A. \]

THE TRANSFORMED TRANSVERSE-FIELD EQUATIONS

The transverse-field equations are written in the following operator form [Braxton and Marcus, 1956]:
\[
(\kappa d(\nabla_j) + i \Gamma_0 \delta^{(0)}(0)) \varphi' = -i \psi'(z) \tag{1}
\]
The transverse field vector \( \varphi' \) and the equivalent transverse current vector \( \varphi'' \) are expressed respectively in terms of \( \psi' \) and \( \psi'' \) through
\[
\varphi' = \Theta_0 \psi', \quad \varphi'' = R(\nabla_j) \psi'' \tag{2}
\]
With the notations given above, the operators \( R(\nabla_j) \) and \( R(\nabla_j) \) become:
\[
R(\nabla_j) = \begin{bmatrix}
1 / \mu_0 \hat{x} \times (1 / \mu) \hat{y} \times & 1 / \mu_0 \hat{x} \times (1 / \mu) \hat{y} \times \\
1 / \mu_0 \hat{x} \times (1 / \mu) \hat{y} \times & 1 / \mu_0 \hat{x} \times (1 / \mu) \hat{y} \times
\end{bmatrix}
\]

In open waveguides with a cross section extending in the region \((-\infty < x < +\infty, -\infty < y < +\infty)\), equation 1 can be usefully represented through a two-dimensional Fourier-integral transform. Consequently the transform of the transverse field vector \( \varphi \) obeys the following equation
\[
\left[ \Theta(\delta / \partial z) - i \kappa k \right] \varphi = -i \Gamma_1 \varphi \tag{5}
\]
where \( \varphi \) is the transform of the equivalent transverse current vector and the operator \( k \) is expressed by
\[
k = \Gamma_1 R(\nabla_j)(\Gamma_1 - i \kappa) \tag{6}
\]
Moreover, the vector \( \varphi \) must satisfy proper boundary conditions at the terminals of the structure. Therefore, a complete operator formulation of the problem at hand is
\[
\begin{align*}
\mathbf{L} \cdot \varphi &= -\mathfrak{d} \\
\varphi \mathfrak{d}(\mathcal{A}) &= \mathfrak{d}(\mathcal{A}) \tag{7}
\end{align*}
\]
where \( \mathbf{L} \) and \( \mathfrak{d} \) can be deduced by inspection of equation 5. The statement \( \mathfrak{d}(\mathcal{A}) \) expresses the boundary conditions on \( \varphi \) in an operator language. In what follows, a stratified medium is considered to be extended between the extremes \( z \) and \( z' \), where \( z < z' \).

The medium may be bounded, that is, \( z_a, z_b, \) and \( z' \) are both finite; in this case, on the boundaries, impedance-type conditions are assumed. When the medium is
considered unbounded, that is, \( z_2 \rightarrow -\infty \) and/or \( z_2 \rightarrow +\infty \), proper radiation conditions must be introduced in defining \( \Theta(z) \).

**THE USE OF THE MATRICANT IN INTEGRATING THE TRANSVERSE-FIELD EQUATION**

The problem of integrating equation 7 can be approached by introducing the matricant \( M(z, z) \) of the system (8), defined by [Pease, 1965]

\[
[\Theta_{\ell}(\partial/\partial z) - ik_{z'}k]M(z, z) = 0
\]

\[
M(z, z) = \Theta, \quad \ell = 0
\]

The matricant of the system has the following properties:

\[
M(z, z) = M(z, z_2)M(z_2, z)
\]

\[
M(z, z) = [M(z, z)]^{-1}
\]

In a source-free region \( z_1 < z < z_2 \), where \( \Theta = 0 \), the matricant \( M(z, z) \) expresses the transfer operator on the wave vector \( \vec{q}(z) \), from section \( z_2 \) to section \( z_1 \) according to

\[
\vec{q}(z) = M(z, z_2)\vec{q}(z_2) \quad \text{for} \quad z_1 < z < z_2
\]

In a thoroughly general case the system matrix \( \mathbf{k} \) may be a function of \( z \). Therefore, the matricant equation (8) can be solved iteratively and the following Peano expansion is obtained:

\[
M(z, z_2) = \Theta + ik_0 \int_{z_2}^{z} k(z') dz' + ik_0 \int_{z_2}^{z} k(z') \int_{z_2}^{z} k(z'') dz'' dz'
\]

\[
- ik_0 \int_{z_2}^{z} k(z') \int_{z_2}^{z} k(z'') \int_{z_2}^{z} k(z''') dz''' dz'' dz'
\]

\[
\ldots
\]

Even if the expansion (11) converges slowly, in principle it gives a valid solution [Pease, 1965]. Moreover, in an homogeneous region, \( \mathbf{k} \) is a constant operator, and the expression (13) assumes the more convenient form:

\[
M(z, z_2) = \exp\{ik_0(z - z_2)\}
\]

Equation 14 can be quickly evaluated, through a series expansion, for slabs which are electrically thin, that is, when \( z - z_2 \) is a few wavelengths long. For thicker slabs the evaluation of the matricant (14) can be again performed through the usual matrix-function expansion on the basis of the eigenvectors of \( \mathbf{k} \). In a stratified system with many slabs, the matricant can be expressed by a repeated use of equations 10 and 14.

The matricant has been used to evaluate the field radiated by a line source [Daniele and Zich, 1970] and by a point source [Zich, 1971] localized in a plane \( z = z_2 \). The field at \( z = z_2 \) and \( z = z_2 \) has been expressed as a function of unknown constants to be evaluated through the jump condition at the source. Even more complicated physical systems, such as a multifluid plasma [Daniele, 1971], can be studied through a matricant formulation.

**GREEN'S FUNCTION AND THE FIELD SCATTERED BY STRATIFIED MEDIA**

The matricant theory gives a direct way to compute the Green's function in anisotropic stratified media and to evaluate the scattered field.

In order to invert equation 7 the Green's function \( g(z, z') \) can be introduced through the following equation:

\[
\vec{q}(z) = -\int_{z'} g(z, z') \vec{q}(z') dz'
\]

The Green's function \( g(z, z') \) and its adjoint as \( g^*(z, z') \) must satisfy the equations

\[
L \cdot g(z, z') = \Theta, \quad \Theta(z - z') \cdot \Theta(z) = \delta(z)
\]

\[
L^* \cdot g^*(z, z') = \Theta, \quad \Theta(z - z') \cdot \Theta(z) = \delta(z)
\]

where the operator \( L \) is the adjoint of \( L^* \). If the adjointness is taken with respect to a symmetric inner product, the operator \( L^* \) becomes

\[
L^* = -\Theta_{\ell}(\partial/\partial z) - ik_{z'}k
\]

and its domain \( \mathbb{D}(L^*) \) is defined by

\[
[\mathbb{D}(L)^{\ast}, \mathbb{D}(L^{\ast})] \quad \text{if} \quad \Theta_{\ell} \cdot \mathbb{D}(L) = \mathbb{D}(L^*)
\]

Moreover, the use of the adjointness relation for the problem at hand yields:

\[
[s^*(z, z')]^* = g(z', z)
\]

As suggested by Bresler and Marcwitz [1966], the evaluation of \( g(z, z') \) can be obtained in terms of the source-free solution both for \( L \) and for \( L^* \). In what follows, two sets of four vectors, \( u_n(z) \) and \( v_n(z) \), solutions of the following wave equations for \( L \) and \( L^* \), will be used:

\[
L \cdot u_n(z) = 0
\]

\[
L^* \cdot v_n(z) = 0
\]
where (i) $u(z)$ and $u_3(z)$ satisfy the boundary conditions of $\mathcal{E}$ for $z = z_0$, $u_3(z)$ and $u_3(z)$ do the same for $z = z_0$, and (ii) $v(z)$ and $v_3(z)$ satisfy the boundary conditions of $\mathcal{E}_3$ for $z = z_0$. From these definitions it follows that the vectors $u_3(z)$ and $u_3(z)$ represent a field due to sources lying in $z' < z$, since they obey the transverse wave-vector equation and meet the boundary conditions at $z = z_0$. Conversely, $u_3(z)$ and $u_3(z)$ represent fields radiated by sources lying in $z' > z$. The vectors $v(z)$ have a similar significance for a transposed medium. With the preceding positions, such a transposed medium is a specular image of the original medium with respect to the plane $z$. However, each of its slabs is defined by the system operator $k^2$. An alternative physical interpretation can be obtained by observing that the vector $v_3(z)$ equals the inner product of the operator $\Gamma$, with the source-free solutions in an alternative transposed medium, each slab of which is defined by a dielectric permittivity $\varepsilon^*$ and a magnetic permeability $\mu^*$. An explicit expansion of $g(z, z')$ in terms of $u_3$ and $v_3$ must take into account the boundary conditions at the boundaries of the structure, both for $z$ and $z'$. Therefore, from equation 20, it appears that $g(z, z')$ may be put in the form

$$g(z, z') = \begin{cases} a_{12} u_3(z) v_3(z') & z > z', \ h, k = 1, 2 \\ a_{21} u_3(z) v_3(z') & z < z', \ p, q = 3, 4 \end{cases}$$

(21)

The summation symbol is omitted as is usual in tensor notation. The coefficients $a_{ij}$ must be determined through the jump condition for $g(z, z')$

$$g(z', z') - g(z', z') = \Theta,$$

(22)

derived by the integration of equation 14 from $z'$ to $z$.

The problem of expressing $u_3(z)$ and $v_3(z)$ can be approached through the use of the matricant. Indeed if $u_3(z)$ is known at a given section $z = z_0$, according to equation 12, it follows that

$$u_3(z) = M(z, z_0) u_3(z_0)$$

(23)

Usually the vectors $u_3(z)$ are known at the terminals $z_0$ $(m = 1, 2)$ and $z_n$ $(m = 3, 4)$, if the structure is bounded. Conversely if the structure extends to $z = -\infty$ and/or to $z = +\infty$, the radiation conditions permit one to write immediately $u_3(z)$ at the first $(m = 1, 2)$ and/or at the last $(m = 3, 4)$ discontinuity plane. Similarly the vectors $v_3(z)$ obey the equation:

$$v_3(z) = M_3(z, z_0) v_3(z_0),$$

(24)

where $M_3(z, z_0)$ is the matricant of the adjoint problem and is defined by

$$\begin{align} \phi^* \cdot M_3(z, z_0) & = 0 \\ M_3(z_0, z_0) & = \Theta \end{align}$$

(25)

The adjoint matricant $M_3(z, z_0)$ is related to $M(z, z_0)$ by

$$M_3(z, z_0) = [M(z_0, z)]^*$$

(26)

Therefore, equation 24 can be rewritten as

$$v_3(z) = v_3(z_0) \cdot M(z_0, z),$$

(27)

In order to satisfy the boundary conditions (ii) we can assume

$$v_3(z_0) = u_3^c(z_0)$$

(28)

where $u_3^c(z_0)$ is the reciprocal set of $u_3(z_0)$, satisfying

$$u_3(z_0) \cdot u_3^c(z_0) = \delta_{zz}.$$  

(29)

Therefore, the vectors

$$u_3^c(z) = v_3(z) = u_3^c(z_0) \cdot M(z_0, z),$$

(30)

give the reciprocal set of $u_3(z)$ in every section $z$ since they obey

$$u_3^c(z) \cdot u_3^c(z_0) = u_3^c(z_0) \cdot u_3^c(z) = \delta_{zz}.$$  

(31)

From the property expressed by equation 31 it follows that $u_3^c(z)$ and $u_3^c(z)$ are orthogonal to both $u_3(z)$ and $u_3(z)$ and, therefore, satisfy the boundary conditions of $\mathcal{E}_3$ for $z = z_0$. Quantities $u_3^c(z)$ and $u_3^c(z)$ do the same at $z = z_0$.

Since the idemfactor $\Theta$, can be expressed in terms of $u_3^c, u_3^c$ by

$$\Theta = u_3^c(z) u_3^c(z)$$

(32)

when $v_3^c(z) = u_3^c(z)$, equation 22 leads to the following identities

$$a_{12} = \delta_{zz}, \ h, k = 1, 2$$

(33)

$$a_{21} = -\delta_{zz}, \ p, q = 3, 4$$

Consequently, the Green's function is

$$g(z, z') = \begin{cases} u_3(z) u_3^c(z') & z > z', \ h = 1, 2 \\ -u_3(z) u_3^c(z') & z < z', \ p = 3, 4 \end{cases}$$

(34)

which can also be put in the following form

$$\begin{align} M(z, z_0) & = u_3(z_0) u_3^c(z_0) \\ M_3(z, z_0) & = -u_3(z_0) u_3^c(z_0) \end{align}$$

(35)
when assuming a reference plane, where \( u_x(z_0) \) and \( u_y(z_0) \) are known. Finally the transverse-wave vector in the natural basis can be obtained by antitransforming \( \varphi(z) \) expressed in the form of (15); that leads to

\[
\varphi(z, \theta) = -\left(\frac{k_z}{4\pi}z\right) \int_0^\infty \varphi(z', \theta) \, dz'
\]

where \( \varphi(z', \theta) \) and \( \varphi(z') \) are now written as \( \varphi(z', \theta) \) and \( \varphi(z') \) to emphasize their dependence on \( \theta \).

The field scattered by a stratified system can be expressed in terms of the matrices after decomposing the total field \( \varphi \) into the incident field \( \varphi' \) and the scattered field \( \varphi'' \). Let us examine structures which extend to \( z = -\infty \). The region lying at the left of the first discontinuity plane \( z = z_2 \) is called region (0) and contains all the sources. In particular it will be assumed that the sources are confined in \( z < z_2 \) with \( z < z_2 \). The total and the scattered fields can be expressed in terms of the incident field through:

\[
\begin{cases}
\varphi(z) = M(z, z_2)u_x(z_2)u_y(z_2)\varphi(z_2) & z \geq z_1, \quad h = 1, 2 \\
\varphi(z) = -M(z, z_2)u_x(z_2)u_y(z_2)\varphi(z_2) & z \leq z_1, \quad p = 3, 4
\end{cases}
\]

where \( z_1 \) is a reference plane such that \( z < z_1 < z_2 \).

When the primary field is the plane wave defined as follows,

\[
\varphi' = \frac{1}{i}A \exp \left[ik(\hat{z}+\hat{r})\right] \quad \hat{r} = \theta \hat{\theta} + \kappa \hat{k},
\]

the total and the scattered field are

\[
\begin{cases}
\varphi(z) = M(z, z_2; k_\perp)u_x(z_2; k_\perp)u_y(z_2; k_\perp)\varphi(z_2; \theta) & m = 1, 2 \\
\varphi(z) = -M(z, z_2; k_\perp)u_x(z_2; k_\perp)u_y(z_2; k_\perp)\varphi(z_2; \theta) & p = 3, 4
\end{cases}
\]

**AN EXAMPLE**

The method previously discussed will now be applied to the system shown in Figure 1. An anisotropic ferrite slab of thickness \( h_0 \) is separated from the perfect conducting ground plane at \( z = 0 \) by means of a dielectric slab of thickness \( h_1 \). The source is a magnetic line current parallel to \( \theta \) and lies at \( z = z_2, x = 0 \). The ferrite slab is polarized with a dc field \( H_0 = H_y \), perpendicular to the line source. The free space extends beyond the ferrite slab from \( z_2 \) to \( z_\infty \).

This problem has been already examined by Daniels and Zieh [1970]. Claricicco and Huckle [1969] have studied a similar problem with the polarizing field parallel to the line source; radiation from a slot on a conducting plane covered by an unbounded magnetoplasma has been considered by Schadri [1968].

In the present example, we can distinguish three different regions with the following properties:

(i) region 1, \( 0 < z < z_1 = h_1 \), a dielectric slab in which

\[
\begin{align*}
(\text{i}) & \quad \varepsilon = \varepsilon_1 \\
(\text{ii}) & \quad \varepsilon = 1
\end{align*}
\]

(ii) region 2, \( z_1 < z < z_2 = (h_1 + h_2) \), a polarized ferrite slab in which

\[
\begin{align*}
(\text{ii}) & \quad \varepsilon = \varepsilon_1 \\
(\text{iii}) & \quad \varepsilon = 1
\end{align*}
\]

In this case the source is in region 1 and is represented by

\[
J_m = J_0 \delta(z - z_2) \delta(x)y
\]

Since the problem is bidimensional, a one-dimensional Fourier transform can be used instead of the bidimen-
sional Fourier transform previously adopted. Therefore, the transformed current vector becomes

\[ \varphi \rightarrow i \int_0^\infty \delta(z - z_0) \delta (z - z_0) \]  
\[ \varphi = \varphi_0 + \int_0^\infty \delta(z - z_0) \delta(z - z_0) \]  
which leads to

\[ \varphi = \varphi_0 + \int_0^\infty \delta(z - z_0) \delta(z - z_0) \]  
\[ \varphi = \varphi_0 + \int_0^\infty \delta(z - z_0) \delta(z - z_0) \]  

The evaluation of the field radiated in region 3

\[ \varphi(x) = -M(z, 0)u_0(x)u^0(0) - M(0, z_0)\varphi_0 \]  
\[ z > z_0, \quad h = 1, 2 \]  

The use of this equation requires the knowledge of the vectors \( u_0(x), u^0(x) \). The plane \( z = 0 \) is a perfectly conducting plane; therefore, by taking into account boundary condition (i) of equation 20 the vectors \( u_0(0) \) and \( u^0(0) \) can be written in the form:

\[ u_0(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  
\[ u^0(0) = \begin{bmatrix} 0 \\ i \end{bmatrix} \]  
\[ u_0(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  
\[ u^0(0) = \begin{bmatrix} 0 \\ i \end{bmatrix} \]  

The choice of \( u_0(x) \) and \( u^0(x) \) requires the use of the radiation conditions for \( z \rightarrow \pm \infty \). Therefore, it is necessary to introduce the basis of \( k \) eigenvectors (plane-wave basis). Let \( \lambda_{1,2}, \xi_1, \) and \( \xi_2 \) be the eigenvalues, the right-hand and the left-hand eigenvectors of \( k \), respectively. In general it is

\[ k = \lambda \xi \xi^* \]  
\[ \lambda_{1,2} = \lambda \xi \xi^* \]  

and the matricant (14) becomes

\[ \exp\{ik_0(z - z_0)\} = \exp\{ik_0 \lambda_{1,2}(z - z_0)\} \xi \xi^* \]  
\[ \exp\{ik_0(z - z_0)\} = \exp\{ik_0 \lambda_{1,2}(z - z_0)\} \xi \xi^* \]  

In an isotropic region \( k \) has only two distinct eigenvalues with a two-fold degeneracy. Such degeneracy can be resolved since \( k \) is semisimple. Therefore, the representations (48) and (49) can again be used. In region 3 we have

\[ \lambda_{1,2} = \lambda_{1,2}(1 - \sigma^3)^{1/2} \]  
\[ \lambda_{1,2} = \lambda_{1,2}(1 - \sigma^3)^{1/2} \]  

where the square root is positive if real and has a positive imaginary coefficient otherwise. From equations 50, 49, and 23 it follows that the boundary conditions at \( z = \pm \infty \), that is, the radiation conditions in region 3, require the wave vector to be a linear combination of \( \lambda_{1,2} \) and \( \lambda_{2,1} \) eigenvectors associated with the double eigenvalues \( \lambda_{1,1} = \lambda_{2,2} \). Consequently, a proper choice of \( u_0(z_0) \) and \( u_0(z_0) \) is

\[ u_0(z_0) = \lambda_{1,2} \xi_1 \]  
\[ u_0(z_0) = \lambda_{2,1} \xi_2 \]  
In this example, region 3 is the free space, and the eigenvectors \( \lambda_{1,2} \xi_1 \) and \( \lambda_{2,1} \xi_2 \) can be expressed by

\[ \lambda_{1,2} \rightarrow \begin{bmatrix} \lambda_{1,2} \\ \frac{1}{2}(1 - \sigma^3)^{1/2} \end{bmatrix} \]  
\[ \frac{1}{2}(1 - \sigma^3)^{1/2} \xi_1 \rightarrow \begin{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}(1 - \sigma^3)^{1/2} \end{bmatrix} \\ \frac{1}{2}(1 - \sigma^3)^{1/2} \xi_1 \end{bmatrix} \]  

The vectors \( u_0(0) \) follow from equations 51 and 12; they are

\[ u_0(z_0) = \lambda_{1,2} \xi_1 \]  
\[ u_0(z_0) = \lambda_{2,1} \xi_2 \]  

The vectors \( u^0(0) \) could be evaluated through a transposed problem. However, according to equation 32, the inversion of the matrix formed by the \( u_0(0) \), is more convenient. The numerical evaluation has been done in the case of a source lying at \( z_s = 0 \). The field in region 3 is expressed through equations 46 and 12 in terms of the field in the plane \( z = z_s \); it follows

\[ \varphi(z) = -M(z, z_s)u_0(z_0)u^0(0)\varphi_0 \]  
\[ z > z_s, \quad h = 1, 2 \]  

Equation 33 can be rewritten in the form

\[ \varphi(z) = c^2 \xi_1 \exp\{ik_0(1 - \sigma^3)^{1/2}(z - z_s)\} \]  
\[ z > z_s, \quad h = 1, 2 \]  

\[ c^2 = -u^0(0)\varphi_0 \]  

and, by antitransforming, the transverse wave vector becomes

\[ \varphi(z, x) = (k_0/2\sigma) \int \left[ c^2 \xi_1 \exp\{ik_0(1 - \sigma^3)^{1/2}(z - z_s)\} \right. \]  
\[ \left. \cdot \exp\{ik_0(1 - \sigma^3)^{1/2}(z - z_s) + \sigma x\} \right] \sigma \right. \]  

The above expression is suitable for saddle-point integration and can be rewritten using the polar coordinates \( \sigma \) and \( a \) introduced in Figure 1. In this way the large parameter \( \kappa L \) appears when considering the far field. Figure 2 presents the original integration path, the two branch points \( \sigma = \pm 1 \), the branch cuts, and the deformed path through the saddle point

\[ \sigma = \sin \alpha \]  

Since the residue contributions of leaky-wave poles
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and 4 show different diagrams of \(|E_x| = |E\cdot\hat{e}|\) and \(|E_y| = |E\cdot\hat{e}|\) versus \(\alpha\) for selected values of \(h_t\) at a frequency of 10 GHz. Radiation patterns are symmetric with respect to \(\alpha = 0\) and present peaks which may be related to leaky-wave modes existing on the structure [Claricoats and Huckle, 1969]. To get numerical results \(M(0, z_t)\) and \(M(z_t, z_d)\) must be computed through the exponential form (14) in order to evaluate \(M(0, z_t)\). These matrices have been computed by using the exponential series. The truncation of the series has been conditioned to a predetermined precision. The precision is easily controlled when slabs are electrically short, i.e., not wider than five wavelengths.

In equation 55, a multivalued function appears under the integral sign. It is to be noted that there are only two branch cuts, namely, the cuts corresponding to the indefinite region. The branch cuts arising from the presence of slabs do not appear in this approach. Indeed, the matricant is not a multivalued function, as is shown by the exponential formulation. Arbel and Felsen [1963] have previously observed that in a stratified system the only significant branch cuts are the ones corresponding to indefinite regions.

CONCLUSIONS

By use of the Bresler and Marcuvitz abstract operator methods, the Green's function in anisotropic stratified media has been evaluated. The Green's function approach usually has a theoretical significance since computations are often too complicated.

Fig. 2. Integration path for far-field evaluation.

Fig. 3. Diagrams of \(E_x\) (left) and of \(E_y\) (right) versus \(\alpha\) for \(h_t = .01\) m and for (a) \(h_t = 0\) m, (b) \(h_t = .005\) m, (c) \(h_t = .01\) m, (d) \(h_t = .02\) m, and (e) \(h_t = .03\) m.
Fig. 4. Diagrams of $E_\alpha$ (left) and of $E_\beta$ (right) versus $\omega$ for $h_a = .01$ m and for (a) $h_a = 0$ m, (b) $h_a = .028$ m, (c) $h_a = .030$, (d) $h_a = .032$ m, and (e) $h_a = .034$ m.

But in the present case the use of a two-dimensional Fourier transform leads to a matrix formulation which straightforwardly applies to the computer. Radiation from a line source in the presence of an anisotropic ferrite slab has been determined through a saddle-point integration. Computer programs have been written for numerical evaluation of the results, and a set of diagrams has been presented. At this point, problems involving electrically short slabs can be successfully handled.

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