GENERALIZED WIENER-HOPF TECHNIQUE FOR WEDGE SHAPED REGIONS OF ARBITRARY ANGLES

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ABSTRACT

A new technique for solving diffraction problems in angular shaped regions is here presented. This technique applies both for impenetrable wedge and penetrable wedges. The functional equations obtained through this technique present different difficulties of solution according to the geometry of the problem. For example for half-planes and impenetrable or isorefractive right wedges we deal with classic matrix W-H equations. On the contrary dealing with arbitrary media or with not right wedges, we have to introduce new functional equations that we call generalized Wiener-Hopf equations. This paper describes some properties of the generalized Wiener-Hopf equations.

INTRODUCTION

The Wiener-Hopf technique is a powerful and general method for solving discontinuity diffraction problems. However usually it is assumed that for wedge shaped regions these geometries (except for particular wedge angles) do not allow Wiener-Hopf equations. The aim of this work is the introduction of the Wiener-Hopf technique also for arbitrary angular wedges and for penetrable regions filled with different media. The obtained results are new functional equations that generalize the classic Wiener-Hopf equations. Our method is based on a generalization of the method used to deduce W-H equations for right angle regions. In particular, through the introduction of oblique coordinates, new equations have been obtained, relating the unilateral Fourier transforms of electromagnetic field in two different directions [1].

By using these equations and by imposing the boundary conditions on the interfaces separating the different homogeneous regions, we can obtain the functional equations relevant to every wedge problem involving not penetrable or penetrable media [1]. In order to give an example in the following a wedge with two face impedances excited by an E-polarized plane wave has been considered (Maliuzhinent problem). Let the axis z of a cartesian coordinates system be parallel to the electric field E (E=Ez) and the axis x coincide with the simmetry axis of the wedge and directed outside. By introducing as unknowns the following unilateral Fourier transforms defined on the positive real semi axis \(x\geq 0, y=0\) or \(\varphi=0\):

\[V_+ (\eta) = \int_0^\infty E_+ (x,0) e^{i\eta x} dx, \quad I_+ (\eta) = \int_0^\infty H_+ (x,0) e^{i\eta x} dx,\]

where \(\omega, k\) and \(\mu\) are the angular frequency, the propagation constant and the permeability in the medium outside the wedge respectively.

The functional equation are [1]:

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\[
M(\eta(\alpha)) = \begin{bmatrix}
V_y(\eta(\alpha)) \\
I_x(\eta(\alpha))
\end{bmatrix} = M(-\alpha) \begin{bmatrix}
V_y(\eta(-\alpha)) \\
I_x(\eta(-\alpha))
\end{bmatrix}
\]

where

\[
M(\eta) = \begin{bmatrix}
\frac{\xi}{\xi - (\xi \cos \Phi + \eta \sin \Phi) z_+ + \omega \mu} \\
\frac{-\alpha z_+ + \omega \mu}{\alpha \cos \Phi + \tau \sin \Phi}
\end{bmatrix}
\]

and \( z_\pm \) are the normal impedances on the boundaries \( \Phi=\pm \Phi \) of the wedges,

\[
\xi = \sqrt{k^2 - \eta^2} \quad \tau = \sqrt{k^2 - \alpha^2} \quad \alpha = \alpha(\eta) = \eta \sin \Phi + \xi \cos \Phi, \eta(-\alpha(\eta)) = \eta \cos 2\Phi - \sqrt{k^2 - \eta^2} \sin 2\Phi.
\]

Eq.1 is a particular case of the functional equations in the Fourier domain, that in general may be written in the following form:

\[
G(\eta)F_x(\eta) = F_y(a(\eta)) + F_z(b(\eta)) + F_t(\eta)
\]

where \( G(\eta) \) (matrix kernel) and \( F_x(\eta) \) (source vector) are known functions depending on the complex variable \( \eta \) and \( F_y, F_z, \) and \( F_t \) are the unknown vectors. The substantial difference with respect to the classic Wiener–Hopf equations is the presence in this functional equation of the known mappings \( a(\eta) \) and \( b(\eta) \). In fact, while the unknown \( F_x(\eta) \) is a classic plus function, i.e. regular in the half-plane \( \text{Im}[\eta] \geq 0 \), in the second member \( F_y(a(\eta)) = F_y(a) \) is a minus function in the a-plane, i.e.regular in the half-plane \( \text{Im}[a] \leq 0 \), and \( F_z(b(\eta)) = F_z(b) \) is a minus function in the b-plane, i.e.regular in the half-plane \( \text{Im}[b] \leq 0 \). We can call the previous equation Generalized Wiener Hopf equation. In some particular cases (half planes or right wedge unpenetrable or isorefractive) we have \( a(\eta)=b(\eta)=\eta \) leading to classic W-H equation that can be solved with the factorization technique.

The major advantages of the introduction of these equations is that they characterize all wedge problems involving both unpenetrable regions and penetrable ones even with anisotropic or bianisotropic media filling them[1].

**ON THE SOLUTIONS OF GENERALIZED WIENER-HOPF EQUATIONS**

Wiener-Hopf technique for solving the classic equations is based on the decomposion and the factorization of generic functions in plus and minus functions (i.e. regular respectively in the half plane \( \text{Im}[\eta] \geq 0 \) and \( \text{Im}[\eta] \leq 0 \). Also dealing with Generalized Wiener-Hopf equations, decomposition and factorization constitute the key for obtaining the solutions. However, whereas the classic decomposition is very easy to obtain, the generalized decomposition is more difficult and in general lead to Fredholm integral equations.

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Decomposition
Let consider a function $F(\eta)$ which is a Fourier transform of a function $f(y)$, and $m(\eta)$ an analytic function of $\eta$ regular on a strip containing the real axis of the plane $\eta$. The generalized decomposition of a function (with respect the mapping $m(\eta)$) consists in getting the two decomposing functions so that:

$$F(\eta) = F_s(\eta) + F_r(\eta)$$

where the functions $F_s(\eta)$ and $F_r(\eta)$ are respectively a plus function in the $\eta$ plane (i.e. regular in half plane $\text{Im}[\eta] \geq 0$) and a minus function in the $m$ plane (i.e. regular in half plane $\text{Im}[m] \leq 0$). If there are not singularities of $F(\eta)$ between the real axis $\text{Im}[\eta]=0$ and the image of $\text{Im}[m]=0$ on the $\eta$ plane (this condition is not restrictive), we obtain the following Fredholm integral equation of the second kind for the decomposing $F_s(\eta)$:

$$F_s(\eta) + \frac{1}{2\pi i} \int_{\gamma} \left( \frac{1}{m(\eta') - m(\eta)} - \frac{1}{\eta' - \eta} \right) F_r(\eta') d\eta' = \frac{1}{2\pi i} \int_{\gamma} \frac{F(\eta')}{m(\eta') - m(\eta)} d\eta'$$

where $\gamma$ runs on the real axis $\eta'$ leaving above the singular point $\eta$. Note that the particular case $m(\eta) = \eta$ leads to the classic Cauchy decomposition formula.

Factorization
Following the same ideas introduced in the classic factorization we can get to the generalized factorization of scalar kernel $G(\eta) = G_s(\eta)G_r(\eta)$, through the logartimic generalized decomposition: $\log[G(\eta)] = \psi_1(m(\eta)) + \psi_2(\eta)$. It yields:

$$G_s(\eta) = e^{\psi_1(m(\eta))}, \quad G_r(\eta) = e^{\psi_2(\eta)}$$

This factorization technique can be extended also to matrix factorization in many cases. In particular it is possible to reduce the generalized factorization of rational matrices to scalar decompositions. This allows to factorize arbitrary matrices with Padé approximations.

Solution of the generalized equations
The process of decomposition and factorization allow us to extend the Wiener-Hopf technique to obtain the solution of the generalized equations:

$$G(\eta)F_r(\eta) = F_r(\eta)G_s(\eta)$$

In this particular case, the generalized factorization of the kernel $G(\eta)$ and the generalized decomposion: $(G_s(m(\eta)))^{-1}F_s(\eta) = G_r(\eta)X_0(m(\eta))$, lead to the solutions:

$$F_s(\eta) = (G_s(\eta))^{-1}X_0(\eta), \quad F_r(\eta) = G_r(m(\eta))X_0(m(\eta))$$

To conclude it must be observed that always we can reduce the arbitrary generalized equation (2) to Fredholm equations. However this procedure sometime requires a process of matrix factorization.

REFERENCES

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