Dyadic Green's Functions in Bounded Media

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Abstract—A new theory for the evaluation of the dyadic Green's functions in a homogeneous medium bounded by perfectly conducting walls is presented. The peculiarities of this theory are 1) the Green's functions are systematically and explicitly decomposed in a singular and an analytical part, and 2) these two parts are formulated in a general manner, i.e., independent on the boundaries. As an application, the Green's functions in a conical guide are evaluated.

I. INTRODUCTION

The technique of the dyadic Green's functions for the solution of electromagnetic boundary-value problems was introduced by Schwinger about 40 years ago. Since then, this subject has been widely explored [1]-[4], and expressions giving the dyadic Green's functions in cylindrical waveguides and cavities have been found by many authors [3]-[7].

The technique of Green's functions is based on the fact that the linearity and the time invariance of Maxwell's equations allow us to write the electromagnetic fields E and H in terms of the electric and magnetic currents J and M by means of the integral representations:

\[ \mathbf{E}(t) = -\int_0^t \mathbf{A}(t', r') \cdot \mathbf{r}' \, dt' - \int_0^t \mathbf{A}(t', r) \cdot \mathbf{M}(t') \, dt' \]  

\[ \mathbf{H}(t) = -\int_0^t \mathbf{A}(t', r) \cdot \mathbf{M}(t') \, dt' - \int_0^t \mathbf{A}(t', r') \cdot \mathbf{M}(t') \, dt' \]

where \( t \) is the observation point, \( t' \) is the source point and \( Z_0, c_0 \), \( V \) represent the dyadic Green's functions of the problem at hand. These functions are not independent and it is possible to express \( T_e, A_e, \) and \( V \) in terms of the dyadic function \( Z \) which will be called dyadic Green's impedance [3].

One of the aspects which will be dealt with in this communication is the behavior of \( Z(t, r') \) when the observation point \( t \) approaches the source point \( t' \). This behavior has been the subject of some discussion in the past. In fact, although it was implicitly recognized that it should be similar to that of the free space [8]-[11], where the dyadic Green's function is represented by

\[ Z(t, r') = \frac{1}{k^2} \int_0^\infty \mathbf{A}(\mathbf{k} \cdot \mathbf{r}) \, e^{j k \mathbf{t} \cdot \mathbf{r}' - j k \mathbf{t} \cdot \mathbf{r}} \, d^3 k \]

the expressions that appeared in the literature didn't show it explicitly. Only a few years ago Johnson et al. [11] obtained a correct decomposition of the dyadic Green's function in its irrotational (longitudinal) part and solenoidal (transverse) part, showing that the irrotational part has a singularity (which is assumed as dominant) of the type:

\[ Z_{0s} = k^2 \mathbf{A}(\mathbf{k} \cdot \mathbf{r}) \, e^{j k \mathbf{t} \cdot \mathbf{r}' - j k \mathbf{t} \cdot \mathbf{r}} \]

However nothing is said on the solenoidal part, which could also be singular (and it fact it is). Moreover this property has been shown only for geometries where the Hara's wave vectors can be introduced.

More recently Daniele [7], starting from an exact expression, valid for \( t = t' \), with pure algebraic manipulations has obtained for homogeneously filled metallic waveguides of arbitrary cross section the following expression:

\[ Z(t, r) = \mu_0 \left( 1 - \frac{1}{k^2} \mathbf{r} \cdot \mathbf{r}' \right) G' + Z_0(t, r') \]

where \( Z_0 \) is a dyadic having components which are analytical (i., regular and infinitely differentiable) in \( t = t' \), and \( G' \) is the scalar Green's function satisfying the equation

\[ (\nabla^2 + k^2)G'(\mathbf{r} - \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \]

from the boundary condition

\[ G' = 0, \quad \text{on the waveguide wall}. \]

In order to obtain a general expression of \( Z \), useful also in structures other than waveguides, a new theory for the formulation of the dyadic Green's function is introduced in this work. This new theory has two peculiarities: 1) the dyadic Green's functions are systematically expressed in a way such that the singular part for \( t = t' \) is put in evidence with respect to the analytical part: and 2) the actual formulation of these two parts is abstract and doesn't depend on the shape of the structure.

As an example, the Green's functions in a conical guide will be evaluated.

II. EVALUATION OF THE DYADIC GREEN'S IMPEDANCE

Let us consider a homogeneous medium with propagation constant \( k \), bounded by a perfectly conducting surface \( S \) with outward normal unit vector \( \hat{n} \). The dyadic Green's function \( Z(t, r') \) satisfies the equation [4]:

\[ \mathbf{H}(t) \times \mathbf{A}(t, r') + \mathbf{A}(t, t) \mathbf{H}(t, r') = 0 \quad \text{in} \quad (t < t') \]

with boundary conditions:

\[ \mathbf{n} \times \mathbf{H}(t, t) = 0, \quad \text{on} \quad S \quad \text{and} \quad t = t \]

If we write \( Z \) as

\[ Z(t, r') = \mu_0 \left( 1 - \frac{1}{k^2} \mathbf{r} \cdot \mathbf{r}' \right) \mathbf{A}(\mathbf{k} \cdot \mathbf{r}) + Z_0(t, r') \]

the following fundamental theorem applies to the decomposition of \( Z \) in its analytical and singular parts.

**Theorem**

If the scalar function \( g(t, r') \) satisfies the Helmholtz equation:

\[ \nabla^2 g + k^2 g = 0 \]

then the dyadic \( Z_0 \) and its derivatives of any order are regular at \( t = t' \). The proof of this theorem is straightforward because, substituting \( \mathbf{g}(\mathbf{t}, \mathbf{r}) \) in (6a), and considering (8a), it can be seen that \( Z_0 \) satisfies the equation:

\[ \nabla \times \nabla \times Z_0 - k^2 Z_0 = -\mu_0 \mathbf{A}(\mathbf{k} \cdot \mathbf{r}) \]

Recalling that, if \( Z \) satisfies (6a), it is \( \mathbf{g} \), we get

\[ \nabla \times \nabla \times Z_0(t, r') = 0. \]

Since \( k \) is assumed locally constant, (9) implies that all components of \( Z_0(t, r') \) are analytical for \( t = t' \), i.e., all derivatives of these components are also regular.
The decomposition of (7) is otherwise arbitrary because \( g(t, r') \) is required only to satisfy (9a) without specific boundary conditions. In particular, \( g \) may be assumed equal to the free space Green's function:
\[
g(t, r') = G_0(t, r') = \exp(-j|t - r'|)/\sqrt{4\pi|t - r'|}. \tag{10}
\]

In this way, for any shape of the surface \( S \), the singular part of the dyadic Green's functions is the same as for free space.

However, the choice indicated in (10) is not the most convenient one, for reasons that will be explained later. It can be seen that it is more convenient to choose \( g(t, r') \) to satisfy the Dirichlet boundary conditions. In this way, we obtain a scalar function which will be denoted with \( G(t, r') \), and will be called scalar Green's function of the first kind. This function, as just stated, must satisfy the Helmholtz equation
\[
(\nabla^2 + k^2)G(t, r') = -\delta(t - r') \tag{11a}
\]
with boundary condition:
\[
G(t, r') = 0, \quad \text{for } r \text{ on } S. \tag{11b}
\]

As a consequence of (11b), if we assume
\[
Z(t, r') = j\omega \mu(k^2 + \nabla^2)G(t, r') + Z_0(t, r') \tag{12}
\]
we may verify, by using the boundary conditions (11b) and (6b), that also \( Z_0(t, r') \) satisfies the boundary condition:
\[
\hat{n} \times Z_0(t, r') = 0, \quad \text{for } r \text{ on } S. \tag{13}
\]

Equation (13), together with the homogeneous Helmholtz equation, is useful in obtaining \( Z_0(t, r') \).

We must observe, however, that (13) and (9) do not define a unique dyadic function \( Z_0 \). An additional condition on \( Z_0 \) is obtained by considering its divergence. From (6a),
\[
k^2 \nabla \cdot Z_0(t, r') = j\omega \mu \nabla \cdot \delta(t - r'). \tag{14}
\]

By computing the divergence of (12) and comparing it with (14), after some algebraic manipulation and using the Helmholtz equation we obtain
\[
\nabla \cdot Z_0(t, r') = -j \omega \mu \nabla \cdot G(t, r'). \tag{15}
\]

In analogy with the proof of the uniqueness of the solution of (6a) with boundary conditions (6b), we may show that (9), (13), and (15) define a unique value for \( Z_0 \). It may be shown that it can be expressed in the form:
\[
Z_0(t, r') = -j \omega \mu [\nabla \cdot F(t, r') - jG(t, r') \times \nabla] \tag{16}
\]
where \( F(t, r') \) and \( H(t, r') \) satisfy the following equations:
\[
\nabla^2 F(t, r') + k^2 F(t, r') = jH(t, r') \tag{17a}
\]
\[
\nabla^2 H(t, r') + k^2 H(t, r') = -jF(t, r') \tag{17b}
\]
with boundary conditions:
\[
F(t, r') = 0, \quad \text{for } r \text{ on } S \tag{17b}
\]
\[
\hat{n} \times (\nabla \times H(t, r')) = 0, \quad \text{for } r \text{ on } S. \tag{18b}
\]

It should be observed that the evaluation of \( G^*, F^* \), and \( H \) involves only the scalar operator \( \nabla^2 \).

III. EVALUATION OF THE FUNCTIONS \( G^*, F^* \), AND \( H \) BY SPECTRAL THEORY OF OPERATORS

Let us consider the scalar problem of finding the eigenfunctions of the operator \( \nabla^2 \):
\[
\nabla^2 \psi_i = \lambda_i \psi_i \tag{19a}
\]
where \( \lambda_i \) is the eigenvalue, and the eigenfunction \( \psi_i \) satisfies the boundary condition:
\[
\psi_i(t) = 0, \quad \text{for } r \text{ on } S. \tag{19b}
\]

The modal index \( i \) is, in fact, a triplet of indexes
\[
i = (m, n, \xi) \tag{20}
\]
because the eigenvector problem is defined in the Hilbert space associated with the three-dimensional geometrical space.

The eigenvectors problem expressed by (19) is self-adjoint; consequently, the eigenfunctions \( \Phi_i \) are orthogonal and satisfy the orthonormality conditions:
\[
\int \psi_i(t) \psi_j(t) \, dV = \delta_{ij}. \tag{21}
\]
The completeness relationship is therefore
\[
\sum_i \psi_i(t) \psi_i(t^*) = 1. \tag{22}
\]

Once \( \Phi_i \) and \( \lambda_i \) are known, the straightforward evaluation of the scalar functions \( G^* \) and \( F^* \) is possible. In the basis of the eigenfunction, the operator \( \nabla^2 \) is in fact expressed by
\[
\nabla^2 = \lambda_i \psi_i \tag{23}
\]
so that from (11a) and (17) we formally obtain the following expressions:
\[
G(t, r') = -\sum_i (\lambda_i + k^2)^{-1} \Phi_i(t) \Phi_i(t^*) \tag{24}
\]
\[
F(t, r') = \sum_i (\lambda_i + k^2)^{-1} \Phi_i(t) \Phi_i(t^*). \tag{25}
\]

In the application of the above formulas it must be noted that, if the volume has points at infinity, one or more of the indexes \( m, n, \xi \) has continuous values. In this case, and for the index becoming continuous, the Kronecker symbol in (21) must be replaced by the Dirac function, and the sums appearing in (22), (24), and (25) by integrals.

For the dyadic function \( H \) the eigenvector problem is the vectorial one, still relevant to the operator \( \nabla^2 \):
\[
\nabla^2 H = \mu H \tag{26a}
\]
where \( H \) fulfills the boundary condition:
\[
\hat{n} \times (\nabla \times H(t)) = 0, \quad \text{for } r \text{ on } S. \tag{26b}
\]
Also this problem is self-adjoint, and the following orthonormality and completeness relationships apply
\[
\int \psi_i(t) \cdot \psi_j(t) \, dV = \delta_{ij}, \tag{27}
\]
\[
\sum_i \psi_i(t) \psi_i(t^*) = 1. \tag{28}
\]
\( H(t, r') \) is expressed in the same manner as \( G^* \) and \( F^* \) because in the basis of the eigenvectors \( H \) the operator \( \nabla^2 \) is diagonal. It is therefore
\[
H(t, r') = \sum_i [\mu(\lambda_i + k^2)]^{-1} \Phi_i(t) \Phi_i(t^*). \tag{29}
\]
The index \( i \) consists of three indexes \( (m, n, \xi) \) which may be
continuous; in this case the same considerations as before apply.

In conclusion, we observe that the expression of $Z$, given in (12), which splits it into its regular and singular parts, can be written in a more compact form by recalling that, from (24) and (25), it can be shown that

$$V'V'r - (1/k^2)VV'G' = -(1/k^2)VV'G_0$$  \hspace{1cm} (30)

where $G_0 = -\Sigma \phi(x) / \delta^r \phi(x)'(r')$ is the static Green's function, satisfying the equation

$$V'G_0 = -\delta(r - r')$$  \hspace{1cm} (31a)

with boundary conditions

$$G_0 = 0, \quad \text{for } r \text{ on } S.$$  \hspace{1cm} (31b)

We obtain in this way the expression, already obtained in [11]:

$$Z = j\omega[\Sigma (-1/k^2)VV'G_0 - \nabla \times H \times V']$$  \hspace{1cm} (32)

which is the decomposition of the dyadic Green's function in its longitudinal, or irrotational ($V'V'G_0$) and transverse, or solenoidal, ($\nabla \times H \times V'$) components.

Equation (32) allows to evaluate more easily $Z$ starting from the static Green's functions $G_0$ and from $H$. However their components are both singular.

IV. CONICAL WAVEGUIDE

A geometry where it is possible to obtain an explicit evaluation of the scalar and vector eigenfunctions of the operator $V'$, satisfying the Dirichlet or the Neumann conditions, is the conical waveguide shown in Fig. 1.

In this section, the case of a homogenous medium is dealt with, while in Fig. 1 the case of a radially stratified medium is shown: the results of this section are easily generalized to account for the stratification.

In this geometry the scalar eigenfunctions $\Phi(\xi)$ appearing in the completeness relationship (22) are given by

$$\Phi(\xi) = f(\xi)\Phi_{m,n}(\theta, \phi)$$  \hspace{1cm} (33)

where

$$f(\xi) = \sqrt{2 \pi \mu \rho(\xi)}$$  \hspace{1cm} (34)

$$\Phi_{m,n}(\theta, \phi) = \frac{1}{2} \left[ \left( 2q_{n+1}\Gamma(q_{n+1}m+1)P_{q_{n+1}}^{m}(-\cos \theta) \right) \left( \frac{(2q_n + 1)\Gamma(q_{n}m+1)P_{q_{n}}^{m}(\cos \theta)}{\sin (q_{n}m+1)\delta \mu \rho \delta_{n,0} P_{q_{n}}^{m}(\cos \theta)} \right) \right]^{1/2}$$  \hspace{1cm} (35)

where $\Gamma$, $\rho$, $P_{q}^{m}$ are, respectively, the gamma, spherical Bessel, and associated Legendre functions. The ranges of the three indexes $l$, $m$, $n$ are the following. $\xi$ is a positive continuous index ($\xi > 0$).

$$\Psi_{m,n}(\theta, \phi) = \frac{1}{2} \left[ \left( 2q_{n+1}\Gamma(q_{n+1}m+1)\delta \mu \rho \delta_{n,0} P_{q_{n}}^{m}(\cos \theta) \right) \left( \frac{(2q_n + 1)\Gamma(q_{n}m+1)\delta \mu \rho \delta_{n,0} P_{q_{n}}^{m}(\cos \theta)}{\sin (q_{n}m+1)\delta \mu \rho \delta_{n,0} P_{q_{n}}^{m}(\cos \theta)} \right) \right]^{1/2}$$  \hspace{1cm} (42b)

$m$ is an integer ($m = 0, \pm 1, \pm 2, \ldots$), and $n$ is a positive integer ($n = 1, 2, \ldots$) associated to the set of the positive values $p_{n}$.

It follows

$$\mu_{l} = \lambda_{n} = -\xi^{2}$$

$$L_{l} = -\frac{1}{\xi} \nabla \Phi_{l}$$  \hspace{1cm} (38a)

$$M_{l} = -\frac{1}{\xi_{m}} \nabla \times (n \Phi_{l})$$  \hspace{1cm} (38b)

$$N_{l} = \frac{1}{\xi_{m}} \nabla \times \nabla \times (n \Phi_{l})$$  \hspace{1cm} (38c)

with Hansen's eigenvectors given by

$$\Phi_{l}(\xi) = f(\xi)\Phi_{m,n}(\theta, \phi)$$

where, being introduced the set $q_{n}(n > 0)$ determined by imposing the condition

$$dP_{q_{n}}^{m}(\cos \theta) = 0$$  \hspace{1cm} (39)

$k_{m}$ and $k_{n}$ are the cutoff constants for the conical guide defined by

$$k_{m}^{2} = p_{n}(p_{n} + 1)$$  \hspace{1cm} (40a)

and the functions $\Psi_{m,n}(\theta, \phi)$ are given by

$$\Psi_{m,n}(\theta, \phi) = f(\xi)\Phi_{m,n}(\theta, \phi)$$

where

$$f(\xi) = \sqrt{2 \pi \mu \rho(\xi)}$$

$$\mu = \lambda_{n} = -\xi^{2}$$

Fig. 1. Geometry of the problem.
and $G'$ and $z_R$ appearing in (12) are given by

$$G' = - \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\Phi_0(r)\Phi_0^*(r)}{k^2 - r^2} \, dr$$

$$z_R = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{L_n(r)M_n^*(r) + M_n(r)L_n^*(r) + N_n(r)N_n^*(r) - I \Phi_0(r)\Phi_0^*(r)}{k^2 - r^2} \, dr.$$  \hspace{1cm} (43)  

(44)

It must be remarked that above integrals can be evaluated explicitly since they involve the following expressions ([3, pp. 14-16], [4, pp. 260-261]):

$$\int_{0}^{\infty} \frac{A(r)rB(r)}{k^2 - r^2} \, dr.$$  \hspace{1cm} (45)

This computation will be omitted here.

V. CONCLUSION

In this communication the properties of the dyadic Green’s functions in homogeneous media bounded by perfectly conducting surfaces have been analyzed. In particular a theory has been developed, evidencing explicitly the contribution which is singular when the observation point approaches the source point. This is expressed in a similar form to that of the free space, allowing a generalization of the theory developed in that case.

REFERENCES


