Estimating Topology of Networks

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(Received 30 June 2006; published 3 November 2006)

We suggest a method for estimating the topology of a network based on the dynamical evolution supported on the network. Our method is robust and can be also applied when disturbances and/or modeling errors are presented. Several examples with networks of phase oscillators, pulse-coupled Hindmarch-Rose neurons, and Lorenz oscillators are provided to illustrate our approach.

DOI: 10.1103/PhysRevLett.97.188701
PACS numbers: 89.75.Hc, 89.75.Fb

Introduction.—The study of complex systems pervades through almost all the sciences, from cell biology to ecology, from computer science to meteorology, to name just a few. A paradigm of a complex system is a network [1] where complexity may come from different sources: topological structure, network evolution, connection and node diversity, and/or dynamical evolution. The macroscopic behavior of a network is determined by both the dynamical rules governing the nodes and the flow occurring along the links. Real networks of interacting dynamical systems—be they neurons, power stations, or lasers—are complex. The research on complex networks has been focused on the properties of the network, such as clustering coefficient, their topological structure as well as on how the topology diversity, and/or dynamical evolution. The macroscopic behavior of a network is determined by both the dynamical rules governing the nodes and the flow occurring along the links. Real networks of interacting dynamical systems—be they neurons, power stations, or lasers—are complex. The research on complex networks has been focused on the their topological structure as well as on how the topology properties of the network, such as clustering coefficient, connectivity distribution, and average network distance, influence its dynamic behavior [2–7]. For example, the effects of these properties on synchronization are well studied in the literature [8–12]. Most networks offer support for various dynamical processes. In this Letter we propose a method for determining the topological structure of a network based on the dynamical evolution supported on the network. The method is robust, can be applied to estimate the connection topology of any subnetwork, and can also be used for “online monitoring” of the dynamic evolution of the network topology.

Consider a network, which is represented by a graph. Recall that a graph is an ordered pair of disjoint sets \((V, E)\) such that \(E\) is a subset of the set of unordered pairs of \(V\). The set \(V\) is the set of vertices and \(E\) is the set of edges. The dynamical evolution on the network is given by:

\[
\dot{x}_i = f_i(x_i, y_i, z_i) + C \sum_{j=1}^{n} a_{ij} h_j(x_j),
\]

where \(i = 1, 2, \ldots, n\), \(x_i = [x_i, y_i, z_i, \ldots]^T \in \mathbb{R}^N\) is the state vector of node \(i\), and \(f: \mathbb{R}^N \to \mathbb{R}^N\) describes the node equations. For simplicity only, here we assume that the first components of each node are connected to each other (more general case will be treated in another paper). Thus, \(h_j(x_i): \mathbb{R} \to \mathbb{R}\) is the output of the node \(j\), and \(C = [1, 0, \ldots, 0]^T\). The topology of the network connections is determined by the adjacency matrix \(A = (a_{ij})\): \(a_{ij} = 1\) if the node \(j\) is connected to the node \(i\), and \(a_{ij} = 0\) otherwise. The Eq. (1) can describe a network of phase oscillators [13], a network of neurons [14], or a network of chaotic oscillators. As an example of a network of phase oscillators, we consider a system of \(n\) phase oscillators,

\[
\dot{\phi}_i = \omega_i + \frac{\kappa}{n} \sum_{j=1}^{n} a_{ij} \sin(\phi_j - \phi_i). \tag{2}
\]

We assume that \(\omega_i\) are normally distributed with mean 0 and variance 1. As an example of a network of neurons, we study a network of pulse-coupled Hindmarch-Rose (HR) neurons, for which the equation of motion is given by:

\[
\dot{x}_i = f_i(x_i, y_i, z_i) + k \sum_{j=1}^{n} a_{ij} (x_j - x_i) + g_i(x_i - V_i)
\]

\[\times \sum_{j=1}^{n} \alpha_{ij} \Gamma(x_j), \tag{3}\]

\[\dot{y}_i = dx_i^3 - y_i, \quad \dot{z}_i = \mu(bx_i + c - z_i),\]

where \(f_i(x) = a x_i^2 - x_i^3 - y_i - z_i\). The matrix \((a_{ij})\) is the adjacency matrix describing the electrical coupling, while the adjacency matrix \((\alpha_{ij})\) describes the synaptic coupling. Finally, as an example of a network of chaotic oscillators, we consider the following array of \(n\) nonidentical Lorenz oscillators:

\[
\dot{x}_i = \sigma_i (y_i - x_i) + c \sum_{j=1}^{n} a_{ij} (x_j - x_i), \tag{4}
\]

\[\dot{y}_i = \rho x_i z_i - y_i, \quad \dot{z}_i = x_i y_i - b z_i,\]
where $i \in V := \{1, 2, \ldots, n\}$, $x_i \in \mathbb{R}$ is the state vector of node $i$, and $f_j : \mathbb{R} \to \mathbb{R}$ describes the node equations. We assume that the mappings $f_j$ and $h_i$ are Lipschitzian for all $i$, that is, there exist positive constants $L_{1i}$ and $L_{2i}$ such that $\|f_j(y_j) - f_j(x_i)\| \leq L_{1i}\|y_j - x_i\|$ and $\|h_i(y_i) - h_i(x_i)\| \leq L_{2i}\|y_i - x_i\|$, for all $i$. Therefore, the tracking error exponentially decays and is ultimately bounded. Since the parameters $k$ and $\varepsilon$ can be freely adjusted, transient performance and the final tracking accuracy are guaranteed. Therefore, $b_{ij} = a_{ij}$ and the system (6) tracks the topology of the network (5). We call Eq. (6) a topology estimator.

Remark 1.—The described method can be applied to any subnetwork. Indeed, let $V_1 \subset V$ be a subset of the vertex set. The outlined procedure can be also used to determine the connections within the subset $V_1$. Assume that for all $j \in V \setminus V_1$, $x_j$ can be measured. Then for all $i \in V_1$ we have

$$\dot{y}_i = f_i(y_i) + \sum_{j \in V \setminus V_1} b_{ij} h_j(y_j) + \Delta_i(y, b_{ij}, t) + u_i,$$

and one can apply the above method.

Remark 2.—In the case when the node is a high-dimensional dynamical system as for the network (1), one can follow the similar steps as for the 1D case. However, the full description of the theory in this case is beyond the scope of the Letter. The network estimator has the form:

$$\dot{x}_i = f_i(x_i) + C \sum_{j=1}^{n} b_{ij} h_j(x_j) + \Delta_i(x, b_{ij}, t) + u_i,$$

where $i, j \in \{1, 2, \ldots, n\}$, $\Delta_i$ represents the unknown nonlinear functions (such as disturbances and modeling errors). We assume that the mappings $f_j$ and $h_i$ are Lipschitzian for all $i$, that is, there exist positive constants $L_{1i}$ and $L_{2i}$ such that $\|f_j(y_j) - f_j(x_i)\| \leq L_{1i}\|y_j - x_i\|$ and $\|h_i(y_i) - h_i(x_i)\| \leq L_{2i}\|y_i - x_i\|$, for all $i$. Therefore, the tracking error exponentially decays and is ultimately bounded. Since the parameters $k$ and $\varepsilon$ can be freely adjusted, transient performance and the final tracking accuracy are guaranteed. Therefore, $b_{ij} = a_{ij}$ and the system (6) tracks the topology of the network (5). We call Eq. (6) a topology estimator.

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Examples.—We now present several examples. In our first example, we consider phase oscillators (2). We investigate standard random symmetric networks, where independently for all connections \( a_{ij} = 1 \) with some probability and \( a_{ij} = 0 \) otherwise. In the numerical simulation presented here \( n = 10 \) and \( \kappa = 0.5 \). Initial conditions for the topology estimator are set to \( b_{ij}(0) = 0.5 \) for all \( i,j \). Figure 1 shows the result of the numerical simulation: \( b_{1,2} \) approaches correctly the value \( a_{1,2} = 0 \) and \( b_{6,2} \) tends to the value \( a_{6,2} = 1 \).

In our next example we consider a network of HR oscillators (3). We assume that the neurons are identical and the synapses are fast and instantaneous. The parameter \( g_s \) is the synaptic coupling strength. The reversal potential \( V_s > x_i(t) \) for all \( x_i \) and all \( t \); i.e., the synapse is excitatory. We set \( V_s = 2 \). The synaptic coupling function is modeled by the sigmoidal function \( \Gamma(x_j) = 1/(1 + \exp(-10(x_j - \theta_s))) \), where \( \theta_s = -0.25 \). In the numerical example presented here: \( a = 2.8, d = 4.4, c = 5, b = 9, \mu = 0.001, g_s = 0.34, k = 0.05, \) and \( n = 10 \). Again the topology estimator correctly estimates the topology of the network, that is the matrices \( (a_{ij}) \) and \( (a_{ij}) \), as illustrated on the Fig. 2.

Finally, as a third example, we consider a network of \( n \) nonidentical Lorenz oscillators (4), for which the values of \( \sigma_j \) are randomly chosen in the interval \([9.2; 9.4]\), and \( \rho = 28, c = 0.1, b = 8/3, \) and \( n = 16 \). For the numerical simulation presented below, we assume that the values of the first raw of the adjacency matrix \( A = (a_{ij}) \) are \( a_{1,j} = 0 \) for \( j = 1, 6, 10, 12, 14, \) and \( a_{1,j} = 1 \) for \( j = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 16 \). Then the system

\[
\dot{x}_i = \sigma_j(\dot{y}_j - \dot{x}_i) + c \sum_{j=1}^{n} b_{ij}(\dot{x}_j - \dot{x}_i) + k(x_i - \dot{x}_i),
\]

\[
\dot{y}_i = \rho \dot{x}_i - \dot{x}_i \dot{z}_i - \dot{y}_i,
\]

\[
\dot{z}_i = \dot{x}_i \dot{y}_i - b \dot{z}_i,
\]

\[
b_{ij} = -\gamma_{ij} c(\dot{x}_j - \dot{x}_i)(\dot{y}_j - x_j),
\]

where \( k \) and \( \gamma_{ij} \) are positives, estimates the elements of the matrix \( A \), that is, \( b_{ij} \to a_{ij} \) as time goes to infinity. Figure 3 shows the estimation of \( b_{ij} \) versus time for \( i = 1 \); for better visual presentation, we show \( b_{ij} + j \) versus time. Note that \( b_{1,5} + 5 \) and \( b_{1,6} + 6 \) approach 6 indicating correctly that \( a_{1,5} = 1 \) and \( a_{1,6} = 0 \).

The proposed approach can be applied for online “monitoring” of the network topology. This implies that the dynamic evolution of the topological structure can be “recorded” by the online “monitor.” Assume, for example, in the above network of \( n = 16 \) nonidentical Lorenz oscillators, we monitor only the fifth and the 11th oscillator. We assume that at \( t = 400 \) there is an abrupt change of the network topology: \( a_{5,11} = a_{11,5} = 1 \) changes to \( a_{5,11} = a_{11,5} = 0 \). Figure 4 shows the result of the numerical simulation of our estimator: \( b_{5,11} \) and also \( b_{11,5} \) estimate correctly the values of \( a_{5,11} \) and \( a_{11,5} \), respectively.
Finally, we present an example of a topology with uncertainty. Consider a network of $n$ nonidentical Lorenz oscillators and assume that we can only measure the variables $x_i$, for $i = 1, 2, \ldots, n_1$. Then the equation for our topology estimator reads:

$$\dot{x}_i = \sigma_j(\dot{y}_i - \dot{x}_i) + c \sum_{j=1}^{n} b_{ij}(\dot{x}_j - \dot{x}_i) + k(x_i - \dot{x}_i) + \Delta_i,$$

$$\dot{y}_j = \rho \dot{x}_i - \dot{x}_j \dot{x}_i - \dot{y}_j, \quad \ddot{x}_i = \dot{x}_i \dot{y}_i - b \dot{z}_i,$$

where $i, j \in \{1, 2, \ldots, n_1\}$, $\gamma_{ij}$ are positives, and $\Delta_i$ represent the effects of the influences of the $n - n_1$ oscillators, that is $\Delta_i = \sum_{n_1 + 1}^{n} b_{ij}\dot{x}_j$. Still we can estimate the connection topology of the $n_1$ oscillators. Figure 5 presents the results of the numerical simulation for the case $n = 17$ and $n_1 = 16$: we plot here the values of $b_{1,j} + j$ versus time for $j = 1, 2, \ldots, 16$ (assuming that $a_{1,1} = a_{1,5} = a_{1,8} = 0$ and $a_{1,j} = 1$ otherwise).

**Conclusions.**—In conclusion, we have suggested a method for estimating the topology of networks. We have shown that the method can be applied to estimate the connection topology in any subnetwork. In addition, we have demonstrated that the approach can be used for online monitoring the dynamic evolution of the network topology. Since our method is robust and works well with disturbances and modeling errors, we think that the method can also be applied to estimate the topology of real (sub)networks. Our approach can be easily generalized to estimate the weights of a weighted network.

This work was partially supported under NSF Grant No. 0327929 (L. K.) and by the National Natural Science Foundation of China under Grant No. 10602026 (D. Y.).

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