Building synchronizable and robust networks

Igor Mishkovski[*], Marco Righero[*], Mario Biey[*] and Ljupco Kocarev[†]
[*]Electronic Engineering Dept., Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129, Torino, Italy
E-mail:{igor.mishkovski, marco.righero, mario.biey}@polito.it
[†]Macedonian Academy of Sciences and Arts, Skopje, Macedonia
E-mail: lkocarev@manu.edu.mk

***Abstract*—In this paper we present a simple, fast, novel algorithm for building networks whose topology has high synchronizability, is robust against failures, and supports efficient communication. We show that the algorithm is able to build these networks in a small number of steps that scales with the networks density. In addition, we track the evolution of different topological properties in the process of generating these networks. The results show that the topological properties are homogeneously distributed and the topology is less authoritative. Furthermore, we show that the networks we generate are more robust than random, geometric random, small-world or scale-free networks with similar average connectivity. Finally, all of the results indicate that the topology of these networks is entangled, which in many cases represents an optimal topology.**

I. Introduction

The model of complex networks permeates our everyday life, due to its simplicity (a certain number of nodes representing individual sites and edges representing connections) and its ability to grasp the essence of many different systems. Commonly cited examples include social networks, technological networks, information networks, biological networks, communication networks, neural networks, ecological networks and other natural and man-made networks. Abundant study of their topology and models is presented in [1]–[3]. An important topic of interest in present research is the collective behavior in complex networks, referring especially to the synchronous state, where all the individual sites operate in unison. The ability of a network to synchronize is commonly referred to as *synchronizability* [4]. This property of the complex networks has many potential uses, such as: finding the optimal topology in order to reach consensus [5], finding optimal topology for communication or transport networks [6], improving the performance of computational tasks [7] and understanding the organizing principles in neural and biological networks [8].

Having in mind the importance of the synchronizability one might ask several questions: how to rewire the network or assign weights to the edges in order to enhance synchronizability? Where to add a small number of edges to improve synchronizability? Which topology is the most synchronizable and how to create it? The scientific community has given hints or answers to some of these questions.

In [9] the authors present a rewiring algorithm, based on simulated annealing, which improves the synchronizability of the network. Another rewiring algorithm, which uses memory tabu search, is proposed in [10]. In [11] the authors propose a weighting procedure, based upon the global structure of network pathways, so as to improve the synchronization in scale-free networks. The authors in [12] use node and edge betweenness for weighting dynamical networks. In [13] Donetti *et al.* propose a stochastic algorithm, based on simulated annealing, for producing entangled networks, i.e. networks with extremely homogeneous structure: with respect to degree, node distance, betweenness and loop distributions. These kinds of networks are characterized by high synchronizability, robustness, efficient communication, etc.

In this paper we are concerned with the issue of creating networks with high synchronizability and robustness by using a simple and fast algorithm. More specifically, given a fixed number of nodes N and an average connectivity $\langle k \rangle$ we built networks with enhanced synchronizability. In addition, we explore some of the topological properties in order to find out more about the obtained network topology. We argue that the topology of the networks created by the algorithm has similar structure and characteristics as the entangled topology. Finally, we compare the vulnerability of the obtained networks with random, geometric, small-world and scale-free topologies by using the measures proposed in [14], [15].

This paper is organized as follows. In Section 2 we summarize the basic results about synchronizability. In Section 3 we give the main motivations behind the algorithm and describe it. Results are given in Section 4, where we also inspect the structural properties of the obtained networks. Additionally, we compare the algorithm with other existing algorithms for creating synchronizable networks, either from scratch or by rewiring. Section 5 presents results related to the vulnerability of the obtained networks. Section 6 concludes this work.

II. A measure of synchronizability

A relevant contribution in determining the stability of the synchronized states was given in [16], [17], by using the eigenvalues of the Laplacian matrix representing the network. We briefly recall the main ideas in the following. Consider a network of N identical dynamical systems with symmetric coupling. The equations of motion for the system are: $\dot{x}_i = F(x_i) + \sigma \sum_{j=1}^{N} L_{ij} H(x_j)$, where $i = 1, ..., N$, $\dot{x} = F(x)$ is the dynamics of each uncoupled individual node, H is the coupling function, σ is the overall coupling strength, and L is the Laplacian matrix. The local stability of the synchronized state $x_i = s, \forall i$ is determined by the corresponding variational equations, which can be diagonalized into N blocks of the form $\dot{y} = [DF(s) + \lambda DH(s)]y$ where y represents different

978-1-4244-5309-2/10/$26.00 ©2010 IEEE

mode of perturbation from the synchronized state. We have $\lambda = \sigma\lambda_i$ for the ith block, where $\lambda_1 = 0 \leq \lambda_2 \leq ... \leq \lambda_N$ are the eigenvalues of L. The master stability function (MSF) [16] for this equation determines the stability of the synchronized state. In particular, the synchronized state is stable if $\Lambda(\sigma\lambda_i) < 0$, for each $i = 2, ..., N$. For large class of chaotic oscillatory systems, there exists a bounded interval (α_1, α_2) on which $\Lambda(\lambda) < 0$. In this case, there exist coupling strengths σ for which the synchronized state is linearly stable, if and only if $\lambda_N/\lambda_2 < \alpha_2/\alpha_1$ [17]. The ratio $Q = \lambda_N/\lambda_2$ depends only on the network topology, while the ratio α_2/α_1 depends on the dynamics in the network. Furthermore, since the interval of the stable synchronized state is larger for smaller Q, thus, one might conclude that smaller Q means more synchronizable network. In practice, the MSF is not always negative only in finite interval (see class-A networks in [18] and the dynamical systems of class Γ_0 and Γ_1 [19]). However, other measures of synchronizability often go "hand in hand" with λ_N/λ_2 [9].

III. Algorithm for building synchronizable networks

In general, random networks have better synchronizability than regular and might have better than small-world networks when they are above their percolation transition [17], [20]. Furthermore, small-world networks have better synchronizability than scale-free networks [21]. In addition, Nishikawa *et al.* discovered that Q decreases when the heterogeneity of some measures of small-world networks declines, even if the average distance increases [22]. In [20] the authors found out that Q is proportional to the betweenness heterogeneity. Hong *et al.* conclude that a small value of the maximum betweenness centrality is an important factor for better synchronizability. The complete correlation between homogeneity and synchronizability for *any connected network* is given in [22]:

$$(1 - \frac{1}{N})\frac{k_{max}}{k_{min}} \leq Q \leq (N-1)k_{max}l^e_{max}D_{max}\langle D\rangle, \quad (1)$$

where N is the number of nodes in the network, k_{min} and k_{max} are the minimum and the maximum degree, respectively, D_{max} is the maximum length of the shortest path between two nodes, l^e_{max} is the maximum normalized edge betweenness and $\langle D\rangle$ is the average length path. Eq. (1) confirms that homogeneous networks have high synchronizability, because in this case k_{max} and l^e_{max} are smaller. However, the combination of small network distances and homogeneous distribution of connectivities and loads makes the network more synchronizable.

The above mentioned consideration helped us in building a simple algorithm capable of reducing Q and able to build undirected and unweighted networks with enhanced synchronizability. Summarizing, the algorithm is based on two premises: *i.* random networks show good synchronizability and *ii.* homogeneous properties make a network more synchronizable (see Eq. (1)).

The only two inputs in the algorithm are the number of nodes N in the network and the average node degree $\langle k\rangle$. The algorithm is the following.

1) $N\frac{\langle k\rangle}{2}$ edges are placed randomly by using the Watts and Strogatz model with rewiring probability $p = 1$ [23].
2) Search for the node n_i which has the maximum degree.
3) Discover all the neighbors of the node n_i, choose the node n_j with the highest degree and delete the edge between the nodes n_i and n_j.
4) Search for two nodes n_k and n_l with the lowest degree, which are not connected together.
5) Place an edge between the nodes n_k and n_l.
6) Stop if an equilibrium is reached, otherwise go to 2.

The first step exploits premise *i*, while the rest of the steps exploit premise *ii*. In addition, if the network become disconnected after removing the edge in the third step, the algorithm searches for the second most connected neighbor of the node n_i, and it removes the edge between these two nodes, and so on. There are slight chances that there is no edge of the node n_i that could be removed without disconnecting the network. If this occurs the algorithm returns to the second step when it chooses the second most connected node, and so on. The algorithm stops when the equilibrium is reached. An equilibrium is reached when the algorithm can not change the network topology, i.e. each iteration of the algorithm alternates the topology between two possible states. As a final result, between the two topologies, the algorithm chooses the one with lowest Q. The good features of the algorithm are: it exploits only local information, i.e. the degree of the node, it is faster and much easier to implement than the algorithm for creating optimal topologies proposed in [13] and the rewiring algorithms proposed in [9] and [10]. However, the algorithm has certain drawbacks which can be noticed by looking at the simulation results and which will be discussed in the following Section.

IV. Simulation results

Using the above mentioned algorithm, we constructed two sets of networks, each one with 50, 100, 200, 300 and 500 nodes and having average node degree 4 and 6 respectively. For each size the number of generated networks is 10. Fig. 1 shows the averaged synchronizability Q for networks with different number of nodes and average node degree (upper panel $\langle k\rangle = 4$, lower panel $\langle k\rangle = 6$) as a function of the algorithm iterations. The comparison of the results for the network with $N = 50$ and $\langle k\rangle = 6$ with the one obtained in [13] shows that we get the same value for Q just after 20 iterations of the algorithm, making the proposed algorithm faster and easier to implement than the one proposed in [13]. On the other hand, we can compare the results for the network where $N = 200$ and $\langle k\rangle = 6$ to the ones presented in [12] and we get a slight worse results (in our case Q is around 10, instead of 6). Of course, here we are addressing different problem, i.e. we want to build networks with enhanced synchronizability from scratch, and the authors in [12] are rewiring an existing topology in order to improve its synchronizability.

For a small-world network with $N = 50$, $\langle k\rangle = 6$, and $p = 0.1$ [23], we computed the average value of Q, which turned

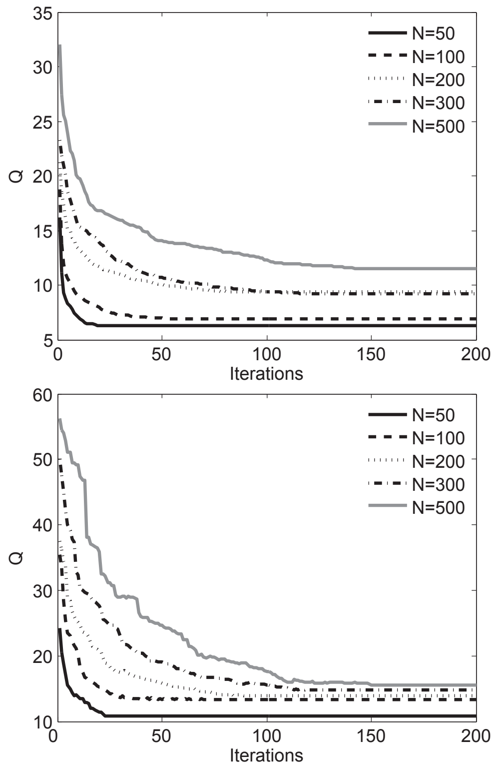

Figure 1. Q as a function of the iterations of the algorithm for networks with different number of nodes N and $\langle k \rangle = 4$ for the upper panel, and $\langle k \rangle = 6$ for the lower panel (data are averaged over 10 networks).

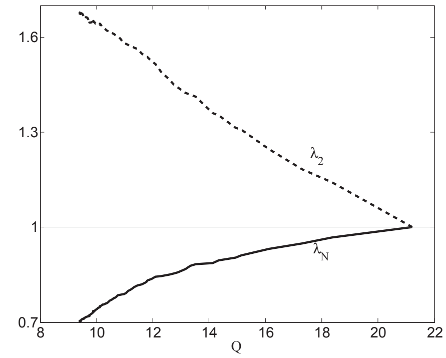

Figure 2. Behavior of the relative λ_2 and λ_N as a function of Q for network with $N = 200$ and $\langle k \rangle = 6$.

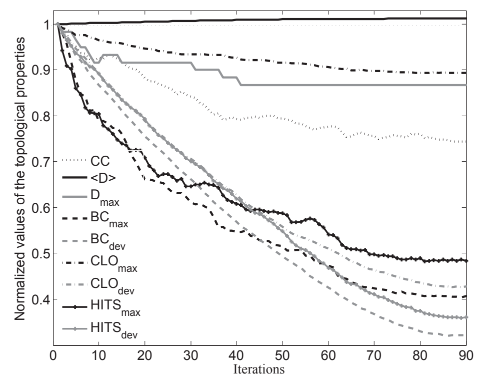

Figure 3. Topological properties of the obtained network with $N = 200$ and $\langle k \rangle = 6$.

out to be around 16, thus from Fig. 1 the synchronizability of the network produced by the algorithm is 2.7 times greater, while in the case of the scale-free network it is greater more than 4 times. The same analysis for the networks with $N = 200$ and $\langle k \rangle = 6$ shows that the algorithm produces networks around 2.1 times better synchronizable than the small-world and around 3.3 times than the scale-free networks.

A drawback of the proposed algorithm, as shown in Fig. 1, is the fact that synchronizability of the obtained network decreases (i.e Q increases) when $\langle k \rangle$ decreases or the number of the nodes in the network N increases. This means that algorithm might not produce a network with high synchronizability if the final aim is to produce sparse network topology. If this is the case, then we suggest to use stochastic algorithms for rewiring or building synchronizable topology, like the ones proposed in [10], [12], [13]. We also performed analysis involving the contribution of λ_2 and λ_N in Q through the iterations of the algorithm. Fig. 2 shows the relative increase of λ_2 and the relative decrease of λ_N with respect to their initial values, i.e. the values they had at the first step of the algorithm, as a function of Q. In this case we analyze network with $N = 200$ and $\langle k \rangle = 6$. It noticeable that the algorithm influences λ_2 more than λ_N, i.e. λ_2 is increased by 70% with the respect to the initial value, and λ_N is decreased by 30%. This is due to the fact that the initial random network is not homogeneous enough [22].

In the next part we want to inspect the obtained networks using some topological properties. In Fig. 3 we present the clustering coefficient (CC), average length path ($\langle D \rangle$), maximum length path (D_{max}), maximum normalized node betweenness (BC_{max}), the standard deviation of the normalized node betweenness (BC_{dev}), the maximum normalized closeness centrality (CLO_{max}), the standard deviation of the normalized closeness betweenness (CLO_{dev}), the maximum authority value using the HITS algorithm [24] ($HITS_{max}$) and the standard deviation of HITS values ($HITS_{dev}$). These measured are normalized with respect to their value obtained from the initial network and are plotted as a function of the iteration steps of the algorithm. Among all values just $\langle D \rangle$ remains almost unchanged (slightly increases) in the resulting network, which correspond to the results shown in [22], whereas all other properties are smaller in the resulting network. The BC_{dev} decreases the most, so that this property can be a good indicator for better synchronizable networks (see also [20]). The standard deviation of the authority rank (represented by $HITS_{dev}$) decrease by around 65%. This supports the idea that the less authoritative the network, the more synchronizable it is. In addition the maximum value of the betweenness centrality decreases by around 60%, which is an argument that this is an efficient topology for communication networks (see [6]). Other good indicators for networks with enhanced synchronizability are the standard deviation of the closeness centrality (it decreases by 55%) and $HITS_{max}$ (it decreases by 50%). These results are totally correlated to Eq. (1) which confirms that the proposed algorithm produces networks with enhanced synchronizability. In order to completely satisfy Eq. (1) the networks should have low value for k_{max} and the ratio k_{max}/k_{min} should be close to 1. The algorithm itself satisfies both the conditions.

The homogeneous structure, the vanishing clustering coefficient and the short average distances are some of the main properties of an entangled (or interwoven) topology [13], topology which is optimal in many senses, such as: synchronization, robustness and support for efficient communication.

V. Vulnerability of the proposed networks

In this Section we show that the obtained networks, besides having enhanced synchronizability, also represent a robust topology. For a vulnerability measure we use the maximal value of the pointwise vulnerability of the network [15] defined as: $V = \max_i \frac{E-E(i)}{E}$. Here E is global network efficiency [14], defined as: $E = \frac{1}{N(N-1)} \sum_{i \neq j} \frac{1}{d_{ij}}$ and $E(i)$ is the network efficiency after removal of the $i-th$ node and all its edges. N is the total number of nodes and d_{ij} is the minimal distance between the $i-th$ and $j-th$ nodes. We compare the vulnerability of the network produced by the algorithm with the: ER model of random network, geometric random network, BA model of scale-free network and WS model of small-world network. The networks had 500 nodes and average connectivity 6. The most robust network is the optimal network obtained from the algorithm ($V = 0.0055$), it was twice more robust than the WS small-world network ($V = 0.0113$) and the ER random network ($V = 0.0125$), 7 times more robust than the BA scale-free network ($V = 0.0371$) and 17 times more robust than the geometric random network ($V = 0.0947$).

VI. Concluding remarks and future work

In conclusion, we have presented a novel algorithm for building networks whose topology has enhanced synchronizability, high robustness, and supports efficient communication (also called entangled networks). The algorithm is fast and very simple compared to others presented in literature. In addition, we have inspected the topological properties in order to give insights about the properties one topology should have in order to be optimal and robust. The conclusion, as in [13], is that entangled structures, i.e very homogeneous structures, and democracy, i.e. low authority rank, are instrumental to get synchronizability and robustness. Finally, we have shown that the networks built with the proposed algorithm are more robust than other networks with similar average connectivity, such as random, geometric random, small-world and scale-free.

As future development, a possible extension to the proposed algorithm is to take into account, at each iteration, the maximum edge betweenness to measure synchronizability and to accept only topologies with lower value of maximum edge betweenness or Q. We have to check the improvement in performance against the additional computational burden given by this modification.

Acknowledgments

M.R. thanks the Istituto Superiore Mario Boella for financial support.

References

[1] Steven H. Strogatz. Exploring complex networks. *Nature*, 410(6825):268-276, March 2001.
[2] R. Albert and A.-L. Barabási, "Statistical mechanics of complex networks," *Reviews of Modern Physics*, vol. 74, no. 1, pp. 47-97, Jan 2002.
[3] M. E. J. Newman, "The structure and function of complex networks," *SIAM Review*, vol. 45, no. 2, pp. 167-256, 2003.
[4] M. Jalili, A. A. Rad, M. Hasler, "Enhancing synchronizability of dynamical networks using the connection graph stability method," *International Journal of Circuit Theory and Applications*, vol. 35, no. 5-6, pp. 611-622, 2007.
[5] S. Barbarossa, G. Scutari, "Decentralized Maximum Likelihood Estimation for Sensor Networks Composed of Nonlinearly Coupled Dynamical Systems," *IEEE Transactions on Signal Processing*, Volume 55, Issue 7, Part 1, July 2007.
[6] R. Guimerà, A. Díaz-Guilera, F. Vega-Redondo, A. Cabrales, and A. Arenas, "Optimal network topologies for local search with congestion," *Physical Review Letters*, vol. 89, no. 24, p. 248701, November 2002.
[7] G. Korniss, M.A. Novotny, H. Guclu, Z. Toroczkai, and P.A. Rikvold, "Suppressing Roughness of Virtual Times in Parallel Discrete-Event Simulations," *Science* 299, 677, 2003.
[8] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences (Cambridge Nonlinear Science Series)*. Cambridge University Press, April 2003.
[9] A. A. Rad, M. Jalili, M. Hasler, "Efficient Rewirings for Enhancing Synchronizability of Dynamical Networks," *Journal of Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 18, no. 3, Sept. 2008.
[10] B. Wang, H.-W. Tang, Z.-L. Xiu, C.-H. Guo, T. Zhou, "Optimization of network structure to random failures," *Physica A*, 368, 607-614, 2006.
[11] M. Chavez, D. U. Hwang, A. Amann, H. G. Hentschel, and S. Boccaletti, "Synchronization is enhanced in weighted complex networks." *Phys Rev Lett*, vol. 94, no. 21, June 2005.
[12] M. Jalili, A. Ajdari Rad, and M. Hasler, "Enhancing synchronizability of weighted dynamical networks using betweenness centrality," *Phys. Rev. E* 78, 016105, 2008.
[13] L. Donetti, P. I. Hurtado, and M. A. Munoz, "Entangled networks, synchronization and optimal network topology," Sep 2005.
[14] V. Latora and M. Marchiori, "Efficient behavior of small-world networks," *Phys. Rev. Lett.* 87, 198701, Oct 2001.
[15] V. Latora and M. Marchiori, "Vulnerability and protection of infrastructure networks," *Physical Review E*, vol. 71, N. 015103, 2005.
[16] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Physical Review Letters*, vol. 80, no. 10, pp. 2109+, March 1998.
[17] M. Barahona and L. M. Pecora, "Synchronization in small-world systems," *Physical Review Letters*, vol. 89, no. 5, 2002.
[18] P. Checco, M. Biey, L. Kocarev, "Synchronization in random networks with given expected degree sequences," *CHAOS, SOLITIONS and FRACTALS*, Vol.35 pp.562-577, 2008.
[19] L. Huang, Q. Chen, Y.-C. Lai, L. M. Pecora, "Generic behavior of master-stability functions in coupled nonlinear dynamical systems," *Physical Review E* 80, 036204, 2009.
[20] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, "Factors that predict better synchronizability on complex networks," *Phys Rev E Stat Nonlin Soft Matter Phys*, vol. 69, no. 6 Pt 2, June 2004.
[21] H. Hong, M. Y. Choi, and B. J. Kim, "Synchronization on small-world networks," *Physical Review E*, vol. 65, no. 2, pp. 026 139+, January 2002.
[22] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, "Heterogeneity in oscillator networks: Are smaller worlds easier to synchronize?" *Physical Review Letters*, vol. 91, no. 1, pp. 014 101+, July 2003.
[23] D. J. Watts, S. H. Strogatz, "Collective dynamics of 'small-world' networks," *Nature 393*, pp. 440-42, 1998.
[24] J. Kleinberg, "Authoritative sources in a hyperlinked environment," *Journal of the ACM* 46 (5): 604632.