Synchronization in Complex Hybrid Networks

Paolo Checco
Dept. of Electronics
Politecnico di Torino
Turin, Italy
Email: paolo.checco@polito.it

Mario Biey
Dept. of Electronics
Politecnico di Torino
Turin, Italy
Email: mario.biey@polito.it

Ljupco Kocarev
Institute for Nonlinear Science
University of California
San Diego, USA
Email: lkocarev@ucsd.edu

***Abstract*— Synchronization in complex hybrid networks is studied. Hybrid networks presents the so called "small world" phenomenon, i.e. small distance between any pair of nodes and clustering effect, and can be considered as the union of a *local* and a *global* graph, the former providing local connections and the latter providing small distances. After recalling a few results stated in previous papers, we prove that although local graph networks do not synchronize when the number N of nodes is large, the addition of only a small number of global edges makes these hybrid networks synchronize. The obtained results are supported by numerical examples.**

I. Introduction

Real networks of interacting dynamical systems – be they neurons, power stations or lasers – are complex. Many real-world networks are small-world [1] and/or scale-free networks [2]. The presence of a power-law connectivity distribution, for example, makes the Internet a scale-free network. The research on complex networks has been focused so far on the their topological structure [3]. However, most networks offer support for various dynamical processes. In this paper we propose to study one aspect of dynamical processes in non-trivial complex network topologies, namely their synchronization behaviors.

The general question of network synchronizability, for many aspects, is still an open and outstanding research problem [4]. In this context, an important contribution has been given by Pecora and Carroll in [5], where, for a network of coupled chaotic oscillators, they derived the so-called *Master Stability Equation* (MSE), and introduced the corresponding *Master Stability Function* (MSF). Consequently, the stability analysis of the synchronous manifold for the network under consideration can be decomposed in two sub-problems [5]. The first sub-problem consists of deriving the MSF for the network nodes, *i.e.* to study in which region of the complex plane the MSE admits a negative largest Lyapunov exponent (LE). The second sub-problem is to verify whether the eigenvalues of the so-called *connectivity matrix* of the network, apart from the zero-eigenvalue, lie in the synchronization region(s). This approach is particularly relevant because the MSE depends only on the nodes local dynamics and on the *coupling matrix*.

In this work we first recall some results reported in previous papers [6], [7], namely that for typical systems only three main scenarios may arise as a function of coupling strength. Then, we study synchronization properties of hybrid networks. These networks possess the so-called small-world phenomenon, shown by many real networks: small distance between any pair of nodes and clustering effect. Hybrid networks can be described by a graph that is a (disjoint) union of a *global* graph, consisting of "long edges" providing small distances, and a *local* graph, consisting of "short edges", which provide local connections. We prove that although local graph networks do not synchronize when the number N of nodes is large, the addition of only a small number of global edges makes these hybrid networks synchronize. The obtained results are supported by numerical examples. Finally, we close our paper with conclusions.

II. Preliminaries

A. Network equations

Let us consider a network with N identical nodes, each being a (chaotic) oscillator. Let $\boldsymbol{x}_i$ be the m-dimensional vector of dynamical variables for the i-th node. Let us assume diffusive coupling. Then, the dynamics of each node is described by:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i) + \sigma \sum_{k=1}^{N} \boldsymbol{D}_{ik}\boldsymbol{x}_k \qquad i = 1, \ldots, N \tag{1}$$

where $\boldsymbol{f} : I\!R^m \to I\!R^m$ describes the oscillator equations, which we assume to admit a chaotic attractor, σ is the overall strength of coupling, while $\boldsymbol{D}_{ik}$ are $m \times m$ real matrixes. Assume that each matrix $\boldsymbol{D}_{ik}$ has the form: $\boldsymbol{D}_{ik} = l_{ik}\boldsymbol{H}$, where l_{ik} is a real number defined in the following and $\boldsymbol{H}$ is a $m \times m$ diagonal matrix, same for all nodes, called *coupling matrix*. The coupling matrix $\boldsymbol{H} = (h_{ij})$ contains the information about which variables are utilized in the coupling and is defined as $h_{ii} = 1$, if the i-th component is coupled, and $h_{ii} = 0$, otherwise. Let $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)^T$, $\boldsymbol{f}(\boldsymbol{x}) = (\boldsymbol{f}(\boldsymbol{x}_1), \ldots, \boldsymbol{f}(\boldsymbol{x}_N))^T$. Furthermore, let the $N \times N$ matrix $\boldsymbol{L} = (l_{ij})$ be the Laplacian matrix, representing the connection topology of the network: $l_{ij} = l_{ji} = -1$ if nodes i and j are connected, $l_{ii} = k_i$ if node i is connected to k_i other nodes, and $l_{ij} = l_{ji} = 0$ otherwise.

Then, we can rewrite Eq. (1) in a more compact form using the direct product of matrixes:

$$\dot{\boldsymbol{x}} = \mathbf{F}(\boldsymbol{x}) + \sigma\,(\boldsymbol{L} \otimes \boldsymbol{H})\,\boldsymbol{x}, \tag{2}$$

where $\mathbf{F}(\boldsymbol{x}) : I\!R^{mN} \to I\!R^{mN}$ is defined as $\mathbf{F}(\boldsymbol{x}) = (\boldsymbol{f}(\boldsymbol{x}_1), \ldots, \boldsymbol{f}(\boldsymbol{x}_N))^T$.

B. Master stability equation and master stability function

The matrix $\boldsymbol{L}$, which will be our main concern, is positive semi-definite, symmetric and has a null eigenvalue in the case of fully connected graphs. Let us denote by $\gamma_1 = 0 < \gamma_2 \leq \ldots \leq \gamma_N$ the eigenvalues of $\boldsymbol{L}$. In particular, γ_2 and γ_N are, respectively, the second and the last eigenvalues of $\boldsymbol{L}$.

Since $\boldsymbol{L}$ is symmetric, the master stability function, in this case, has the form [5]

$$\dot{\zeta} = [\boldsymbol{J}_f + \alpha\, \boldsymbol{H}]\, \zeta, \tag{3}$$

where $\alpha \in I\!R$ and $\boldsymbol{J}_f$ is the Jacobian matrix of $\boldsymbol{f}(\boldsymbol{x})$. Therefore, in this case the corresponding largest Lyapunov exponent or MSF, $\Lambda(\alpha)$, depends only on one parameter, α. The master stability function determines the linear stability of the synchronized state; in particular, the synchronized state is stable if all the eigenvalues of the matrix $\boldsymbol{L}$ are in the *synchronization region* $\mathcal{S} \subseteq I\!R$ where $\Lambda(\alpha) < 0$.

C. Synchronization regions

Discussions in [7] show that for the system (2) the synchronization region $\mathcal{S}$ may have one of the following forms:

- $\mathcal{S}_1 = \emptyset$
- $\mathcal{S}_2 = (\alpha_m, +\infty)$
- $\mathcal{S}_3 = \bigcup_j (\alpha_m^{(j)}, \alpha_M^{(j)})$

Examples of the these scenarios are given in [6], [7], [8]. In the majority of cases α_m, $\alpha_m^{(j)}$, and $\alpha_M^{(j)}$ turn out be positive and, furthermore, in the case $\mathcal{S}_3$ there is only one parameter interval $(\alpha_m^{(j)}, \alpha_M^{(j)})$ on which $\Lambda(\alpha) < 0$. For this reason, we will limit ourself to consider only such cases, focusing, in the remaining of this paper, on the scenarios $\mathcal{S}_2 = (\alpha_m, +\infty)$ and $\mathcal{S}_3 = (\alpha_m, \alpha_M)$. It is easy to see that for $\mathcal{S}_2$ the condition of stable synchronous state is $\sigma\gamma_2 > \alpha_m$. For $\mathcal{S}_3$, one can easily show that there is a value of the coupling strength σ for which the synchronization state is linearly stable, if and only if $\gamma_N/\gamma_2 < \alpha_M/\alpha_m$. Therefore, for a large class of (chaotic) oscillators there exist two classes of networks:

1. Class-A networks: networks whose synchronization region is of type $\mathcal{S}_2$, for which the condition of stable synchronous state is $\sigma\gamma_2 > a$;
2. Class-B networks: networks whose synchronization region is of type $\mathcal{S}_3$, for which this condition reads $\gamma_N/\gamma_2 < b$;

where $a = \alpha_m$ and $b = \alpha_M/\alpha_m$ are constants that depend on $\boldsymbol{f}$, the synchronous state $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \ldots = \boldsymbol{x}_N$ and the matrix $\boldsymbol{H}$, but not on the Laplacian matrix $\boldsymbol{L}$. For typical oscillators $b > 1$.

D. Classical random networks

The primary model for the classical random graphs is the Erdös-Rényi model [9], in which each edge is independently chosen with probability q for some given $0 < q < 1$. In what follows we will denote a classical random graph on N vertices by $G(N, q)$.

E. Random power-law networks

We consider a random model introduced recently by Chung and Lu [10], [11], which produces graphs with a given *expected* degree sequence w. Therefore, this model does not produce a graph with exact given degree sequence, as in the case of the configuration models [12]-[14] or evolution models [2].

Let us denote with $w = (w_1, w_2, \ldots, w_N)$ the expected degree sequence, where w_i is the degree assigned to vertex v_i. The edges are chosen independently and randomly according to the vertex degrees as follows. The probability p_{ij} that there is an edge between v_i and v_j is proportional to the product $w_i w_j$ where i and j are not required to be distinct. There are possible loops at v_i with probability proportional to w_i^2, i.e.

$$p_{ij} = \frac{w_i w_j}{\sum_k w_k} = \rho w_i w_j \tag{4}$$

with $\max_i w_i^2 < \sum_k w_k$ so that $p_{ij} \leq 1$ for all i and j.

May be interesting to note that a classical random graph $G(N, q)$ (see Subsection II-D) on N vertices and edge density q is just a random graph with uniform expected degree sequence $(qN, qN, \ldots, qN)$.

In this paper we will denote the Chung-Lu model of power-law random graph as $M(N, \beta, d, m)$, where N is the number of vertices, $\beta > 2$ is the power of the power law describing the degree sequence, d is the expected average degree, defined as $d = \sum w_i/N$, and m is the expected maximum degree, such that $m^2 = o(Nd)$.

III. Synchronization in hybrid networks

It has been observed that many realistic networks possess the so-called small world phenomenon, with two distinguishing properties: small distance between any pair of nodes, and the clustering effect, i.e. two nodes are more likely to be adjacent if they share a neighbor. In this Section, we consider a hybrid graph model proposed by Chung and Lu [15], which has both aspects of the small world phenomenon. Roughly speaking, a hybrid graph is a union of a global graph $\mathcal{G}_G$ (consisting of "long edges" providing small distances) and a local graph $\mathcal{G}_L$ (consisting of "short edges" respecting local connections).

A. Local graphs

We will consider the local graph to be a grid graph, defined as in [16], with an even maximum vertex degree $\Delta = 2d$, and with a diameter D, function of the number of vertices N on the order of $\mathcal{O}(\sqrt[d]{N})$. Note that paths and cycles turn out to be particular cases of a grid.

Theorem 3.1: When $N \to \infty$ local (grid) graphs for both class-A and class-B networks do not synchronize.

Proof: It is know that, see for example [16], in the case of a grid

$$\gamma_2 < \frac{2d\ln(N-1)}{2(D-2) - \ln(N-1)},$$

if $2(D-2) - \ln(N-1) > 0$. Therefore, $\gamma_2 \to 0$ as $N \to \infty$ for the grid graphs. On the other hand, $2d = \Delta(G) \leq \gamma_N \leq 2\Delta(G) = 4d$. Therefore, $\gamma_N/\gamma_2 \to \infty$ as $N \to \infty$. ∎

B. Global graphs

For the global graph $\mathcal{G}_G$, we consider two cases: classical random graph model $G(N, q)$, described in Subsection II-D, and power-law random graph model $M(N, \beta, d, m)$, described in Subsection II-E.

For any two vertices v_i and v_j, the probability of choosing an edge $v_i v_j$ between v_i and v_j is denoted by $p(v_i, v_j)$, defined as follows:

- $p(v_i, v_j) = 1$ if $v_i v_j$ is an edge of $\mathcal{G}_L$;
- $p(v_i, v_j) = q$ for a classical random graph;
- $p(v_i, v_j) = \rho w_i w_j$ for a power-law random graph.

C. Hybrid network

Let now consider a hybrid network for which equation of the motion can be written as:

$$\dot{\boldsymbol{x}} = \mathbf{F}(\boldsymbol{x}) + \sigma \left[(\boldsymbol{L}_L + \boldsymbol{L}_G) \otimes \mathbf{H} \right] \boldsymbol{x}, \tag{5}$$

where $\boldsymbol{L}_L$ and $\boldsymbol{L}_G$ are the matrixes describing respectively the local graph $\mathcal{G}_L$ and the global graph $\mathcal{G}_G$. Let $N_{total} = N(N-1)/2$ be the possible total number of edges in a network with N nodes and N_L be the total number of local edges. Then $N_G = N_{total} - N_L$ is the number of all possible global edges. Let pN_G, where $0 \leq p \leq 1$, be a number of global edges.

Theorem 3.2: Assume N is large enough and let $\mathcal{G}_G$ be a global graph (classical random graph model or power-law model). Then for class-A networks, given a, there exist a number p, such that $\sigma_c(p) \ll \sigma_c(0)$, where $\sigma_c(p) = a/\gamma_2(p)$, $\sigma_c(0) = a/\gamma_2(0)$, $\gamma_2(p)$ is the second eigenvalue of the matrix $\boldsymbol{L}_L + \boldsymbol{L}_G$, and $\gamma_2(0)$ is the second eigenvalue of the matrix $\boldsymbol{L}_L$. For class-B networks, given $b > 1$, there exist a number p, such that $\gamma_N(p)/\gamma_2(p) < b$, where $\gamma_2(p)$ and $\gamma_N(p)$ are the second and the N-th eigenvalue, respectively, of the matrix $\boldsymbol{L}_L + \boldsymbol{L}_G$.

Proof: Since for $p = 1$ the matrix $\boldsymbol{L}_L + \boldsymbol{L}_G$ is fully connected, it follows that $\gamma_i(1) = N$, $i \geq 2$; hence $\gamma_2(1) = N$ and $\gamma_N(1)/\gamma_2(1) = 1$. On the other hand, on average, $\gamma_2(p)$ is a monotonically increasing function of p and $\gamma_N(p)/\gamma_2(p)$ is a monotonically decreasing function of p [17]. Thus, for both classes of networks (class-A and class-B), there exists a critical value of p, p_c, such that for $p > p_c$, almost all networks (5) are synchronizable. ■

D. Examples

We now present an example. Let the local graph $\mathcal{G}_L$ be a circle and $N = 1200$. It is easy to compute that $\gamma_2(0) = 8.3513 \times 10^{-9}$ and $\gamma_N(0)/\gamma_2(0) = 1436156.321$. Now let us consider two different cases.

(i) Assume that the global graph $\mathcal{G}_G$ is a classical random graph model. Consider first class–A oscillators for which $a = 1$ and $\sigma \leq 10$. Since $\sigma\gamma_2 \ll 1$, the local network $\mathcal{G}_L$ of 1200 oscillators does not synchronize. Consider now the hybrid graph $\mathcal{G}_L \bigcup \mathcal{G}_G$. The dependence of $\gamma_2(p)$ on p is shown in Figure 1(a). It follows that the hybrid graph synchronizes if $\gamma_2(p) > a/\sigma = 0.1$. From the magnified inset

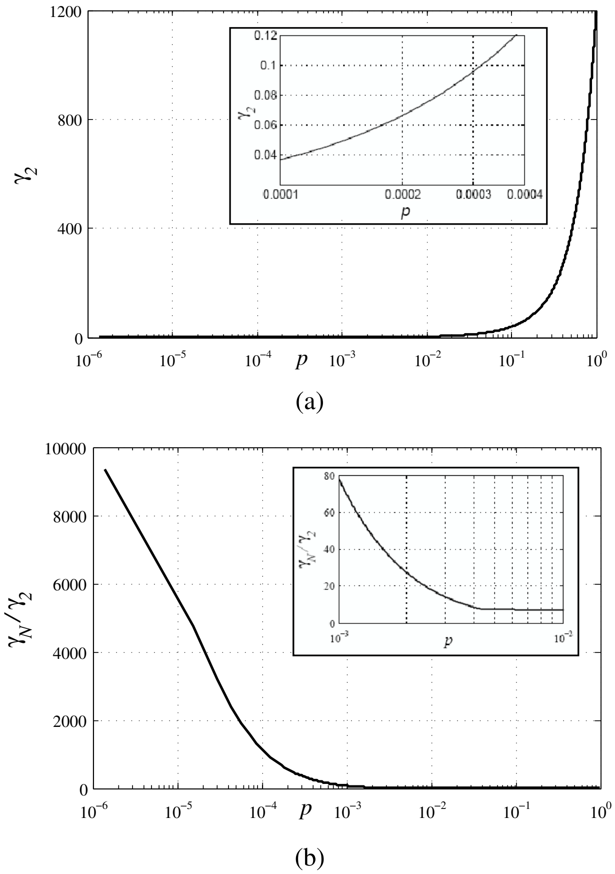

Fig. 1. γ_2 (a) and γ_N/γ_2 (b) versus p for the hybrid model with $N = 1200$. The local graph is a circle and the global graph is a classical random graph.

in Fig. 1(a) we get $\gamma_2(p) > 0.1$ already for $p = 33.30 \cdot 10^{-4}$. We consider now a network of class–B oscillators for which $b = 40$. Since $\gamma_N(0)/\gamma_2(0) \gg 40$, the local network $\mathcal{G}_L$ does not synchronize. Now let us consider the hybrid graph. The dependence of $\gamma_N(p)/\gamma_2(p)$ on p is shown in Fig. 1(b). Since the condition for synchronization is $\gamma_N/\gamma_2 < b$, it follows that the hybrid graph synchronizes for $p = 15.78 \cdot 10^{-4}$. Again this value can be approximately read from the magnified inset in Fig. 1(b). Therefore, adding only a small number of global edges makes the oscillators synchronize.

(ii) Assume now that the global graph $\mathcal{G}_G$ is a random power-law graph. Numerically we consider the graph generated in the following way. First, we choose c nodes at random from all N nodes with equal probabilities and assign them to be *centers*. Second, we add global edges by connecting one node chosen at random from all N nodes to another node randomly chosen from the c centers. Third, when all centers are fully connected with other nodes, we start uniformly to add links between the rest of the nodes. The dependence of $\gamma_2(p)$, $\gamma_N(p)$, and $\gamma_N(p)/\gamma_2(p)$ on p for such model is shown in Figure 2, for $c = 5$. From this figure and several numerical experiments, not reported here for lack of space, we may conclude:

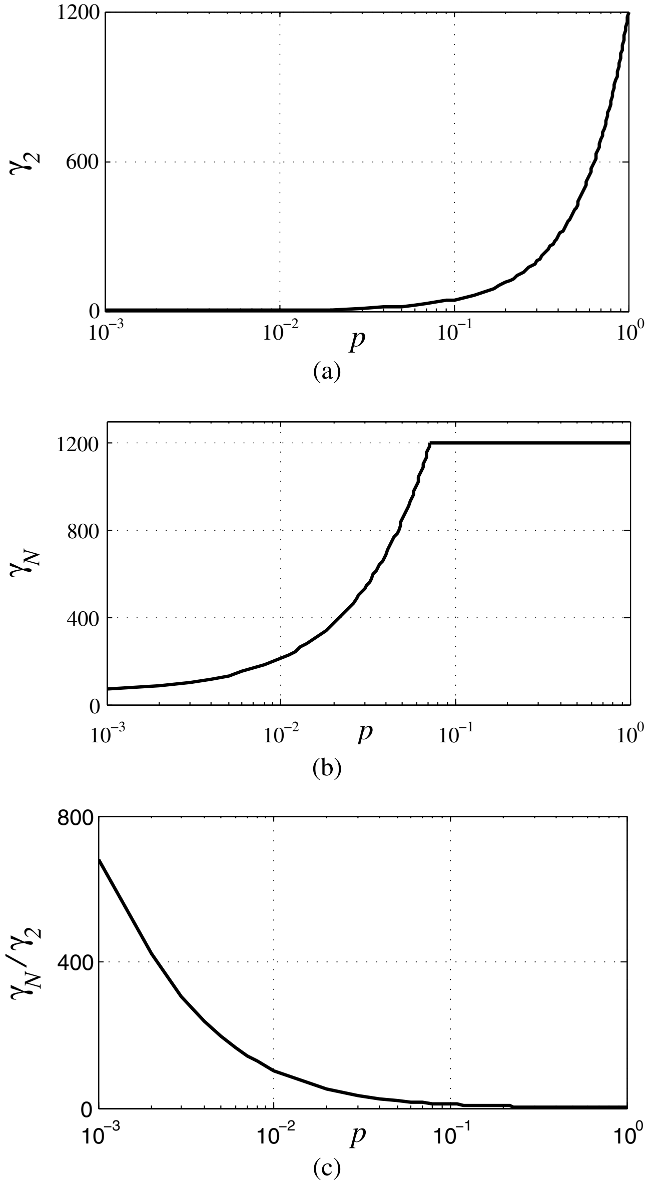

Fig. 2. γ_2 (a), γ_N (b), and γ_N/γ_2 (c) versus p for the hybrid model with $N = 1200$ and $c = 5$. The local graph is a circle and the global graph is a power-law graph.

(a) $\gamma_N(p)$ reaches the maximum value N for smaller value of c; thus, γ_N reaches the value N in the fastest way for $c = 1$;
(b) γ_2 is not affected by c.

Therefore, the random model with c centers only influences synchronization property of class–B networks: if one adds global edges using the model with centers, the network is more difficult to synchronize.

Thus, for example, from Fig. 2(c) it turns out that a class–B network with $b = 40$ will synchronize for $p = 26.70 \cdot 10^{-3}$. This value is higher than the value $15.78 \cdot 10^{-4}$ obtained in the case (i) previously considered. Saying in another way, if the global edges are added independently (i.e. $c = 0$), then the synchronization is optimal.

IV. Conclusion

In this paper we studied synchronization in hybrid networks. First we recalled some results published elsewhere. Namely, for a large class of oscillators two classes of networks exist, for which the condition for stable synchronous state can be explicitly given in terms of the coupling strength σ, of the second and of the last eigenvalues of the Laplacian matrix $\boldsymbol{L}$ of the graph describing the topology of the network, and of constants that depend on local dynamics, synchronous state and the coupling matrix, but not on $\boldsymbol{L}$.

Then we considered hybrid networks, i.e. networks described by a graph that can be considered the union of a local graph and a global graph. We proved that although local graph networks do not synchronize when the number N of nodes is large, the addition of only a small number of global edges makes these hybrid networks synchronize. The obtained results were supported by numerical examples.

Acknowledgement

This work was supported in part by Ministero dell'Istruzione, dell'Università e della Ricerca under PRIN Project no. 2006093814_003.

References

[1] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks", *Nature*, vol. 393, pp. 440 – 442, 1998.
[2] A.–L. Barabási and R. Albert, "Emergence of scaling in random networks", *Science*, vol. 286, pp. 509 – 512, 1999.
[3] R. Albert and A.–L. Barabási, "Statistical mechanics of complex networks", *Rev. Mod. Phys.*, vol. 74, pp. 47 – 97, 2002.
[4] S. Strogatz, *Sync: The Emerging Science of Spontaneous Order*, Hyperion, New York, 2003.
[5] L. Pecora and T. Carroll, "Master stability functions for synchronized coupled systems," *Physical Review Letters*, vol. 80, no. 10, pp. 2109 – 2112, 1998.
[6] P. Checco, L. Kocarev, G. M. Maggio, and M. Biey, "On the Synchronization Region in Networks of Coupled Oscillators," Vol. IV, pp. 800 – 803, ISCAS, 2004.
[7] P. Checco, M. Biey, and L. Kocarev, "Synchronization in random networks with given expected degree sequences", *Chaos, Solitons and Fractals*, 2006, DOI: 10.1016/j.chaos.2006.05.063.
[8] T. Stojanovski, L. Kocarev, U. Parlitz, and R. Harris, "Sporadic driving of dynamical systems," *Phys. Rev. E*, vol. 55, pp. 4035 – 4048, 1997.
[9] P. Erdös and A. Rényi, "On random graphs," *Publ. Math Debrecen*, vol. 6, pp. 290 – 291, 1959.
[10] W. Aiello, F. Chung and L. Lu, "A random graph model for massive graphs," Proceedings of the Thirty–Second Annual ACM Symposium on Theory of Computing, pp. 171 – 180, 2000.
[11] F. Chung and L. Lu, "Connected Components in Random Graphs with Given Expected Degree Sequences," *Annals of Combinatorics*, vol. 6, pp. 125 – 145, 2001.
[12] E. A. Bender and E. R. Canfield, "The Asymptotic Number of Labelled Graphs with Given Degree Sequences," *J. Combinat. Theory* (A), vol. 24, pp. 296 – 307, 1978.
[13] B. Bollobás, *Random Graphs*, Cambridge University Press, 2nd edition, Cambridge, 2001.
[14] M. Molloy and B. Reed, "The Size of the Giant Component of a Random Graph with a Given Degree Sequence," *Combin. Probab. Comput.*, vol. 7, no. 3, pp. 295 – 305, 1998.
[15] F. Chung and L. Lu, "The small world phenomenon in hybrid power law graphs," in *Complex Networks*, E. Ben–Naim, H. Frauenfelder, and Z. Toroczkai, Eds., Springer–Verlag, pp. 91 – 106, 2004.
[16] C. W. Wu, "Synchronization in Arrays of Coupled Nonlinear Systems: Passivity, Circle Criterion, and Observer Design," *IEEE Trans. Circuits Syst. I*, vol. 48, pp. 1257 – 1261, 2001.
[17] M. Fiedler, "Algebraic connectivity of graphs," *Czech. Math. J.*, vol. 23, no. 98, pp. 298 – 305, 1973.