ON THE EXISTENCE OF STABLE EQUILIBRIUM POINTS IN CELLULAR NEURAL NETWORKS

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ABSTRACT

Cellular neural networks are dynamical systems, described by a large set of coupled nonlinear differential equations. The equilibrium point analysis is an important step for understanding the global dynamics and for providing design rules. We yield a set of sufficient conditions for the existence of at least one stable equilibrium point. Such conditions give rise to simple constraints, that extend the class of CNN, for which the existence of a stable equilibrium point is rigorously proved. In addition, they are suitable for design and easy to check, because they are directly expressed in term of the template elements.

1. INTRODUCTION

Cellular neural networks (CNNs) are analog dynamic processors, that have found several applications for the solution of complex computational problems [1, 2]. A CNN can be described as an array of identical nonlinear dynamical systems (called cells), that are locally interconnected. In most applications the connections are specified through space-invariant templates.

CNNs are modeled by large systems of coupled nonlinear differential equations, that have been mainly studied through extensive computer simulations. As far as the dynamic behavior is concerned, CNNs can be divided in two main classes: stable CNNs, with the property that each trajectory (with the exception of a set of measure zero) converges towards an equilibrium point; unstable CNNs, that exhibit at least one attractor, that is not a stable equilibrium point. Due to the complex CNN mathematical model, so far a complete characterization of the two classes above is not available [3].

A preliminary step for investigating CNN dynamics is the equilibrium point analysis: in fact the existence of at least one stable equilibrium point is a necessary condition for the CNN stability, whereas the absence of stable equilibria is a sufficient condition for instability. CNN equilibrium points have been studied in several papers, that have provided important conditions, concerning the existence of stable equilibrium points [4, 5]. Most of these contributions, however, also apply to general networks and do not really exploit the two main characteristics of a CNN, i.e. the local connectivity and the space-invariance structure.

In this paper we provide a set of sufficient conditions, ensuring the existence of at least one stable equilibrium point. Such conditions are different from those reported in the literature and extend the class of CNNs for which the existence of a stable equilibrium point is rigorously proved. In addition they are expressed in term of the template elements: hence they are very easy to check and to exploit for CNN design.

2. SPACE-INVARIANT CNNS

We consider CNNs composed by $N \times M$ cells arranged on a regular grid. We denote the position of a cell with two indexes (k, l) with the assumption that cell $(1, 1)$ is located in the upper left corner and cell (N, M) is located in the lower right corner. The network dynamics is governed by the following normalized state equations

$$\begin{aligned} \dot{x}_{kl} &= -x_{kl} + \sum_{|n|\le r, |m|\le r} A_{nm}\, y_{k+n,l+m} \\ &\quad + \sum_{|n|\le r, |m|\le r} B_{nm}\, u_{k+n,l+m} + I \end{aligned} \quad (1)$$

where x_{kl} and u_{kl} represent the state-voltage and the input voltage of cell (k, l); y_{kl} is the output voltage, defined through the following piecewise linear expression:

$$y_{kl} = f(x_{kl}) = \frac{1}{2}\left(|x_{kl} + 1| - |x_{kl} - 1|\right) \quad (2)$$

Finally r denotes the neighborhood of interaction of each cell; $\boldsymbol{A}$ and $\boldsymbol{B}$ are linear templates, that are assumed to be space-invariant and I is the bias term. The description of the structure is completed by the specification of the boundary conditions, that we assume to be null.

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$C_{NW} \rightarrow \sum_{n=0}^{n=1}\sum_{m=0}^{m=1} A_{nm} > 1$	$C_{N} \rightarrow \sum_{n=0}^{n=1}\sum_{m=-1}^{m=1} A_{nm} > 1$	$C_{NE} \rightarrow \sum_{n=0}^{n=1}\sum_{m=-1}^{m=0} A_{nm} > 1$
$C_{W} \rightarrow \sum_{n=-1}^{n=1}\sum_{m=0}^{m=1} A_{nm} > 1$	$C_{0} \rightarrow \sum_{n=-1}^{n=1}\sum_{m=-1}^{m=1} A_{nm} > 1$	$C_{E} \rightarrow \sum_{n=-1}^{n=1}\sum_{m=-1}^{m=0} A_{nm} > 1$
$C_{SW} \rightarrow \sum_{n=-1}^{n=0}\sum_{m=0}^{m=1} A_{nm} > 1$	$C_{S} \rightarrow \sum_{n=-1}^{n=0}\sum_{m=-1}^{m=1} A_{nm} > 1$	$C_{SE} \rightarrow \sum_{n=-1}^{n=0}\sum_{m=-1}^{m=0} A_{nm} > 1$

Table 1. *Set of conditions $\mathcal{C}$*

$P_{11,nm} \rightarrow C_{NW}$	$P_{1l,nm}\ (l \neq 1, M) \rightarrow C_N$	$P_{1M,nm} \rightarrow C_{NE}$
$P_{k1,nm}\ (k \neq 1, N) \rightarrow C_W$	$P_{kl,nm}\ (k \neq 1, N; l \neq 1, M) \rightarrow C_0$	$P_{kM,nm}\ (k \neq 1, N) \rightarrow C_E$
$P_{N1,nm} \rightarrow C_{SW}$	$P_{Nl,nm}\ (l \neq 1, M) \rightarrow C_S$	$P_{NM,nm} \rightarrow C_{SE}$

Table 2. *Set of conditions $\mathcal{E}$*

3. EQUILIBRIUM POINTS

For the sake of simplicity, we assume that the input and the bias terms are null; however we remark that the results presented in the paper also applies, with slight modifications, to the case of constant inputs and non-zero boundary conditions. In the following, with the term *saturation region* we indicate a linear region of the state space where all the output voltages y_{kl} are saturated (i.e. $\forall\ k,\ l$: $|y_{kl}| > 1$). A saturation region will be denoted by a matrix, containing as entries the output voltage values. We also assume that the template $\boldsymbol{A}$ is represented by a 3×3 matrix (as required in most applications) and that its central element is greater than one (i.e. $A_{00} > 1$), to ensure that the stable equilibrium points are located in saturation regions [1]. We have:

$$\boldsymbol{A} = \begin{bmatrix} A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ A_{0,-1} & A_{00} & A_{01} \\ A_{1,-1} & A_{10} & A_{11} \end{bmatrix} \qquad (3)$$

Under the above assumptions, the $N \times M$ cell CNN turns out to be described by the following state equations:

$$\dot{x}_{kl} = -x_{kl} + \sum_{|n|\leq r, |m|\leq r} A_{nm}\, y_{k+n,l+m} \qquad (4)$$

The results presented in the paper are based on four propositions, that for lack of space are reported without proof.

Proposition 1: Let $S = \{y_{kl},\ (1 \leq k \leq N,\ 1 \leq l \leq M)\}$ be a saturation region. Let us consider the space-variant CNN (SVCNN), associated to (4) with respect to the saturation region S, and described by the following equations:

$$\dot{w}_{kl} = -w_{kl} + \sum_{|n|\leq r, |m|\leq r} P_{kl,nm}\, z_{k+n,l+m} \qquad (5)$$

where $z_{kl} = f(w_{kl})$, according to (2), and $P_{kl,nm}$ is a feedback space-variant template, defined as:

$$P_{kl,nm} = y_{kl}\, A_{nm}\, y_{k+n,l+m} \qquad (6)$$

A sufficient and necessary condition in order that the CNN (4) presents an equilibrium point in the saturation region S is that the SVCNN (5) exhibits an equilibrium point in the saturation region $S' = \{z_{kl}$: $(\forall\ k, l,\ z_{kl} = 1)\}$.

We denote with $\mathcal{C}$ the set of 9 conditions, regarding the 3×3 $\boldsymbol{A}$ template (3), that are reported in Table 1.

Proposition 2: A SVCNN described by the state equation (5) presents an equilibrium point in the saturation region $S' = \{z_{kl}$: $\forall\ k, l,\ z_{kl} = 1\}$ if and only if the space-variant template (6) fulfills the set of constraints $\mathcal{E}$, defined in Table 2.

Proposition 3: A space-invariant CNN, described by equation (4) and template (3), exhibits at least one stable equilibrium point, if and only if there exists a saturation region $\mathcal{S}$, such that the corresponding SVCNN (described by (5) and (6)) satisfies the set of constraints $\mathcal{E}$ defined in Table 2.

The necessity part of the above condition is complex to verify, because it would require to examine all the possible saturation regions. We will concentrate on the sufficient part of the condition and restrict our attention to the saturation regions $\mathcal{S}$, whose output can be written as $y_{kl} = h_k^h h_l^v$, with $h_k^h,\ h_l^v \in \{-1, 1\}$. The space variant template (6) admits of the following expression:

$$P_{kl,nm} = h_k^h h_l^v\, A_{nm}\, h_{k+n}^h h_{l+m}^v \qquad (7)$$

The above transformation can be seen as the result of the applications of two operators, that act on the column and on

$\mathcal{H}^s[\mathcal{V}^r(\boldsymbol{A})] \rightarrow C_{NW}$	$\forall i,\ \mathcal{H}^i[\mathcal{V}^r(\boldsymbol{A})] \rightarrow C_N$	$\mathcal{H}^t[\mathcal{V}^r(\boldsymbol{A})] \rightarrow C_{NE}$
$\forall j,\ \mathcal{H}^s[\mathcal{V}^j(\boldsymbol{A})] \rightarrow C_W$	$\forall i,j,\ \mathcal{H}^i[\mathcal{V}^j(\boldsymbol{A})] \rightarrow C_0$	$\forall j,\ \mathcal{H}^t[\mathcal{V}^j(\boldsymbol{A})] \rightarrow C_E$
$\mathcal{H}^s[\mathcal{V}^u(\boldsymbol{A})] \rightarrow C_{SW}$	$\forall i,\ \mathcal{H}^i[\mathcal{V}^u(\boldsymbol{A})] \rightarrow C_S$	$\mathcal{H}^t[\mathcal{V}^u(\boldsymbol{A})] \rightarrow C_{SE}$

Table 3. *Conditions for the existence of a stable equilibrium point, according to Proposition 4.*

the row respectively of the template $\boldsymbol{A}$. These two operators, called horizontal and vertical, will be denoted with $\mathcal{H}^k$ and $\mathcal{V}^k$ respectively. They are defined as follows:

$$\mathcal{H}^k[\boldsymbol{A}] = \begin{bmatrix} p_k^h\ A_{-1,-1} & A_{-1,0} & q_k^h\ A_{-1,1} \\ p_k^h\ A_{0,-1} & A_{00} & q_k^h\ A_{01} \\ p_k^h\ A_{1,-1} & A_{10} & q_k^h\ A_{11} \end{bmatrix}$$

$$\mathcal{V}^l[\boldsymbol{A}] = \begin{bmatrix} p_l^v\ A_{-1,-1} & p_l^v\ A_{-1,0} & p_l^v\ A_{-1,1} \\ A_{0,-1} & A_{00} & A_{01} \\ q_l^v\ A_{1,-1} & q_l^v\ A_{10} & q_l^v\ A_{11} \end{bmatrix} \quad (8)$$

where $p_k^h = h_{k-1}^h h_k^h$, $q_k^h = h_k^h h_{k+1}^h$, $p_l^v = h_{l-1}^v h_l^v$, and $q_l^v = h_l^v h_{l+1}^v$.

By use of (7) and (8) one obtains:

$$\boldsymbol{P}_{kl} = \mathcal{V}^l\{\mathcal{H}^k[\boldsymbol{A}]\} = \mathcal{H}^k\{\mathcal{V}^l[\boldsymbol{A}]\} \quad (9)$$

Due to the fact that $h_k^h,\ h_l^v \in \{-1, 1\}$ and hence $p_k^h,\ p_k^v,\ q_k^h,\ q_k^v \in \{-1, 1\}$ only four different forms are admissible for the operators $\mathcal{H}^k$ and $\mathcal{V}^l$. For the sake of the simplicity, such forms are denoted by removing from $\mathcal{H}^k$ and $\mathcal{V}^l$ the apexes k and l and by adding two indexes, corresponding to the values of $p_k^h,\ q_k^h$ and $p_k^v,\ q_k^v$ respectively. As an example, the expressions of $\mathcal{H}_{-1,1}$ and $\mathcal{V}_{1,-1}$ are:

$$\mathcal{H}_{-1,1}[\boldsymbol{A}] = \begin{bmatrix} -A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ -A_{0,-1} & A_{00} & A_{01} \\ -A_{1,-1} & A_{10} & A_{11} \end{bmatrix}$$

$$\mathcal{V}_{1,-1}[\boldsymbol{A}] = \begin{bmatrix} A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ A_{0,-1} & A_{00} & A_{01} \\ -A_{1,-1} & -A_{10} & -A_{11} \end{bmatrix} \quad (10)$$

Owing to (8) it is derived that two consecutive operators $\mathcal{H}^k = \mathcal{H}_{ab}$ and $\mathcal{H}^{k+1} = \mathcal{H}_{cd}$ must satisfy the constraint $b = c$. According to this rule, the possible sequences of horizontal operators are defined through a suitable oriented graph, that contains $\mathcal{H}$ as nodes and such that $\mathcal{H}_{ab} \rightarrow \mathcal{H}_{cd}$ iff $b = c$. The same property holds for the operators $\mathcal{V}^k$. In order to give a compact characterization of such sequences we will introduce the following definition.

Definition : given an oriented graph, containing n nodes, $a_1, \dots a_n$, such that $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \dots \rightarrow a_{n-1} \rightarrow a_n \rightarrow a_1$, the corresponding circular sequence is denoted by $\mathcal{C}(a_1, a_2, \dots a_n)$.

Owing to the above definition, it is easily derived that a circular sequence is not altered if the argument is shifted, i.e. $\mathcal{C}(a_1, a_2, \dots a_n) = \mathcal{C}(a_i, a_{i+1}, \dots a_n, a_1, a_2, \dots a_{i-1})$. With the notation $[\mathcal{C}(a_1, a_2, \dots a_n)]^p$ we denote the sequence obtained by iterating p times the circular sequence $\mathcal{C}(a_1, a_2, \dots a_n)$; if $p = 0$ this denotes the null sequence.

The set of all the admissible circular sequences for the horizontal and vertical operators can be expressed as:

$$\begin{aligned} \mathcal{H} &\rightarrow \mathcal{C}([\mathcal{H}_{-1,1}]^m,\ [\mathcal{H}_{1,1}]^p,\ [\mathcal{H}_{1,-1}]^m,\ [\mathcal{H}_{-1,-1}]^q) \\ \mathcal{V} &\rightarrow \mathcal{C}([\mathcal{V}_{-1,1}]^m,\ [\mathcal{V}_{1,1}]^p,\ [\mathcal{V}_{1,-1}]^m,\ [\mathcal{V}_{-1,-1}]^q) \end{aligned} \quad (11)$$

with $(m = 0,\ q = 0,\ p > 0)$ or $(m = 0,\ p = 0,\ q > 0)$ or $(m = 1,\ p \geq 0,\ q \geq 0)$.

The following proposition holds:

Proposition 4 : Let $\mathcal{C}(\mathcal{H}^1, \mathcal{H}^2, \dots \mathcal{H}^p)$ and $\mathcal{C}(\mathcal{V}^1, \mathcal{V}^2, \dots \mathcal{V}^q)$ be two admissible circular sequences of horizontal and vertical operators, respectively. Let a CNN be described by the template $\boldsymbol{A}$. If there exist s, r, t, and u such that the set of conditions reported in Table 3 are satisfied, then there exist N and M such that the CNN exhibits at least one stable equilibrium point.

The application of Proposition 4 to the admissible circular sequences shown in (11) allows to distinguish 6 possible classes of sequences for both the $\mathcal{H}$ and the $\mathcal{V}$ operators. They are reported in the upper part of Table 4 and denoted with $\mathcal{S}_{Hi}$ and $\mathcal{S}_{Vi}$. Each one of the six $\mathcal{S}_{Hi}$ sequences gives rise to a set of possible values for the parameters I, L and K. Table 4 shows that the total number of possible choices for the parameters I, L are 40; then each choice corresponds to one or more values of K. By considering that some cases are incorporated into others, the actual number can be reduced to 16. The same consideration is valid for the six $\mathcal{S}_{Vi}$ sequences, with respect to the parameters J, M and G.

Hence Proposition 4 yields the following procedure: I) for each one of the possible $16 \times 16 = 256$ sets of parameters I, L, and J, M derived from the upper part of Table 4, check the constraints reported in the lower part of Table 4, for all the prescribed values of K and G; II) if such constraints are verified for at least one of the 256 considered cases, then the CNN exhibits a stable equilibrium point. We remark that the above procedure simply requires to check

Case	Sequence	Conditions
S_{H1}	$(m=0,\ p>0,\ q=0)\ \rightarrow\ \mathcal{C}([\mathcal{H}_{11}]^p)$	$I,L,K\ =\ (1,1)$
S_{H2}	$(m=0,\ p=0,\ q>0)\ \rightarrow\ \mathcal{C}([\mathcal{H}_{-1,-1}]^p)$	$I,L,K\ =\ (-1,-1)$
S_{H3}	$(m=1,\ p=0,\ q=0)\ \rightarrow\ \mathcal{C}(\mathcal{H}_{-1,1},\ \mathcal{H}_{1,-1})$	$I,L\ =\ (-1,1)$ or $(1,-1)$ $K\ =\ (-1,1)$ and $(1,-1)$
S_{H4}	$(m=1,\ p>0,\ q=0)\ \rightarrow\ \mathcal{C}(\mathcal{H}_{-1,1},[\mathcal{H}_{1,1}]^p,\mathcal{H}_{1,-1})$	$I,L\ =\ (1,-1)$ or $(-1,1)$ or $(1,1)$ $K\ =\ (1,-1)$ and $(-1,1)$ and $(1,1)$
S_{H5}	$(m=1,\ p=0,\ q>0)\ \rightarrow\ \mathcal{C}(\mathcal{H}_{-1,1},\mathcal{H}_{1,-1},[\mathcal{H}_{-1,-1}]^q)$	$I,L\ =\ (1,-1)$ or $(-1,1)$ or $(-1,-1)$ $K\ =\ (1,-1)$ and $(-1,1)$ and $(-1,-1)$
S_{H6}	$(m=1,\ p>0,\ q>0)\ \rightarrow\ \mathcal{C}(\mathcal{H}_{-1,1},[\mathcal{H}_{1,1}]^p,\mathcal{H}_{1,-1},[\mathcal{H}_{-1,-1}]^q)$	$I,L\ =\ (1,-1)$ or $(-1,1)$ or $(1,1)$ or $(-1,-1)$ $K\ =\ (1,-1)$ and $(-1,1)$ and $(-1,-1)$ and $(-1,-1)$
S_{V1}	$(m=0,\ p>0,\ q=0)\ \rightarrow\ \mathcal{C}([\mathcal{V}_{11}]^p)$	$J,M,G\ =\ (1,1)$
S_{V2}	$(m=0,\ p=0,\ q>0)\ \rightarrow\ \mathcal{C}([\mathcal{V}_{-1,-1}]^p)$	$J,M,G\ =\ (-1,-1)$
S_{V3}	$(m=1,\ p=0,\ q=0)\ \rightarrow\ \mathcal{C}(\mathcal{V}_{-1,1},\ \mathcal{H}_{1,-1})$	$J,M\ =\ (-1,1)$ or $(1,-1)$ $G\ =\ (-1,1)$ and $(1,-1)$
S_{V4}	$(m=1,\ p>0,\ q=0)\ \rightarrow\ \mathcal{C}(\mathcal{V}_{-1,1},[\mathcal{V}_{1,1}]^p,\mathcal{V}_{1,-1})$	$J,M\ =\ (1,-1)$ or $(-1,1)$ or $(1,1)$ $G\ =\ (1,-1)$ and $(-1,1)$ and $(1,1)$
S_{V5}	$(m=1,\ p=0,\ q>0)\ \rightarrow\ \mathcal{C}(\mathcal{V}_{-1,1},\mathcal{V}_{1,-1},[\mathcal{V}_{-1,-1}]^q)$	$J,M\ =\ (1,-1)$ or $(-1,1)$ or $(-1,-1)$ $G\ =\ (1,-1)$ and $(-1,1)$ and $(-1,-1)$
S_{V6}	$(m=1,\ p>0,\ q>0)\ \rightarrow\ \mathcal{C}(\mathcal{V}_{-1,1},[\mathcal{V}_{1,1}]^p,\mathcal{V}_{1,-1},[\mathcal{V}_{-1,-1}]^q)$	$J,M\ =\ (1,-1)$ or $(-1,1)$ or $(1,1)$ or $(-1,-1)$ $G\ =\ (1,-1)$ and $(-1,1)$ and $(-1,-1)$ and $(-1,-1)$

$\mathcal{H}_I[\mathcal{V}_J(\boldsymbol{A})]$	$\rightarrow$	C_{NW}	$\mathcal{H}_K[\mathcal{V}_J(\boldsymbol{A})]$	$\rightarrow$	C_N	$\mathcal{H}_L[\mathcal{V}_J(\boldsymbol{A})]$	$\rightarrow$	C_{NE}
$\mathcal{H}_I[\mathcal{V}_G(\boldsymbol{A})]$	$\rightarrow$	C_W	$\mathcal{H}_K[\mathcal{V}_G(\boldsymbol{A})]$	$\rightarrow$	C_0	$\mathcal{H}_L[\mathcal{V}_G(\boldsymbol{A})]$	$\rightarrow$	C_E
$\mathcal{H}_I[\mathcal{V}_M(\boldsymbol{A})]$	$\rightarrow$	C_{SW}	$\mathcal{H}_K[\mathcal{V}_M(\boldsymbol{A})]$	$\rightarrow$	C_S	$\mathcal{H}_L[\mathcal{V}_M(\boldsymbol{A})]$	$\rightarrow$	C_{SE}

Table 4. *Conditions for the application of Proposition 4 to the admissible $\mathcal{H}$ and $\mathcal{V}$ sequences.*

some sets of inequalities, expressed in term of the template elements; hence it exploits both the local connectivity and the CNN space-invariant structure. We have also verified, by comparison with [4, 5], that the procedure allows to extend the class of CNNs for which a rigorous proof of the existence of a stable equilibrium point is available.

4. CONCLUSION

We have investigated the properties of stable equilibrium points in space-invariant CNNs. We have yielded a set of sufficient conditions for the existence of at least one stable equilibrium point. Such conditions presents two main characteristics: a) they exploit both the CNN local connectivity and the space-invariance structure and hence they are directly expressed in terms of the template elements; b) they are different from the results reported in the literature [4, 5] and allow to extend the class of CNN, for which the existence of a stable equilibrium point is rigorously proved.

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