COMPLEX DYNAMICS IN CELLULAR NEURAL NETWORKS

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ABSTRACT

The occurrence of complex dynamic behavior (i.e bifurcation processes, strange and chaotic attractors) in autonomous space-invariant cellular neural networks (CNNs) is investigated. Firstly some sufficient conditions for the instability of CNNs are provided; then some classes of unstable template are identified. Finally it is shown that unstable CNNs often exhibit complex dynamics and for a case study the most significant bifurcation processes are described. It is worth noting that most CNN implementations exploit space-invariant templates and so far no example of complex dynamics has been shown in autonomous space-invariant CNNs.

1. INTRODUCTION

Cellular neural networks (CNNs) are analog dynamic processors, that have found several applications for the solution of complex computational problems [1, 2, 3]. A CNN can be described as an array of identical nonlinear dynamical systems (called cells), that are locally interconnected. In most applications the connections are specified through space-invariant templates.

The mathematical model of a CNN consists in a large set of coupled nonlinear differential equations, that, apart from small networks, have been mainly studied through extensive computer simulations.

CNN stability was widely investigated and several sufficient conditions for their stability (i.e the convergence of each trajectory towards an equilibrium point) were presented: they are reviewed and summarized in [4].

For what concerns unstable CNNs (i.e. CNNs which exhibit at least one attractor that is not a stable equilibrium point) several examples were shown of CNNs presenting limit cycles [5, 6]. The main properties and characteristics of such limit cycles were studied in a quite effective way through the introduction of space-time spectral techniques [7, 8].

Complex dynamics in CNNs (i.e. networks presenting non-periodic, possibly strange, attractors) have been so far observed in four cases: a) non-autonomous networks composed of two cells [9]; b) autonomous CNNs described by space-variant templates and composed of three cells [10]; c) state-controlled CNNs, with space-variant connections [11]; d) delayed CNNs [12]. To the authors' knowledge, no example of complex dynamics has been shown in autonomous CNNs described by space invariant templates.

The importance of investigating the complex dynamic behavior of autonomous space-invariant CNNs relies on the fact that VLSI implementations exploit an autonomous space-invariant model. Hence the identification of chaotic dynamics in such networks might open the possibility of developing, on the existing CNN chips, innovative chaos-based applications.

In this paper we firstly yield some sufficient conditions, ensuring that a space-invariant CNN be unstable: such conditions present the advantage of being checked by simply looking at the template elements. Then we verify that, for the unstable templates, besides the occurrence of limit cycles, it is quite common to observe bifurcation phenomena, leading to strange and chaotic attractors. Finally we choose a case study and we investigate the most significant bifurcation processes.

2. SPACE-INVARIANT CNNS

We consider autonomous CNNs composed by $N \times M$ cells arranged on a regular grid and denote the position of a cell with two indexes (k, l). We assume that the CNN is described by the original model introduced in [1], whose dynamics turns out to be very similar to that of the model adopted for VLSI implementations (see [3]).

The network dynamics is governed by the following normalized state equations

$$\begin{aligned}\dot{x}_{kl} &= -x_{kl} + \sum_{|n|\le r, |m|\le r} A_{nm}\, y_{k+n,l+m} \\ &\quad + \sum_{|n|\le r, |m|\le r} B_{nm}\, u_{k+n,l+m} + I_{kl} \qquad (1)\end{aligned}$$

where x_{kl} and u_{kl} represent the state voltage and the constant input voltage of cell (k, l); y_{kl} is the output voltage,

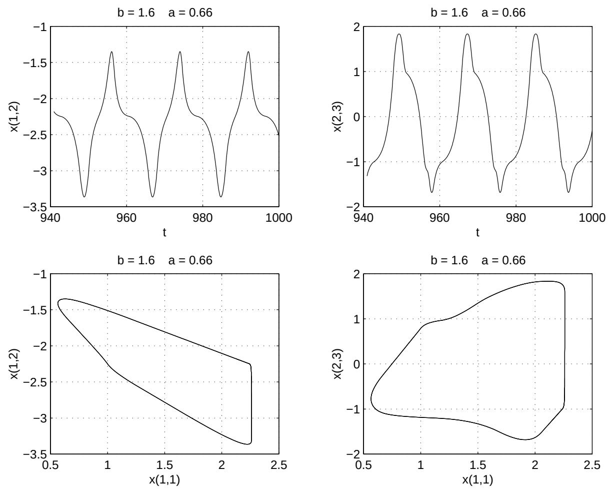

Figure 1: *Period-I limit cycle in a CNN composed by 9 cells and described by the template (4) with* $b = 1.6$ *and* $a = 0.66$.

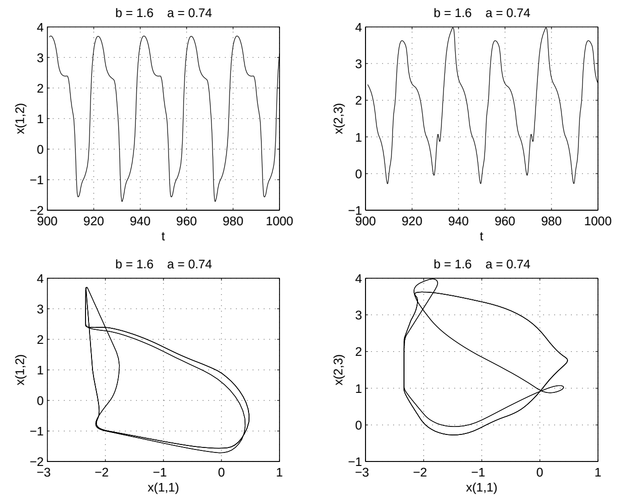

Figure 2: *Period-II limit cycle in a CNN composed by 9 cells and described by the template (4) with* $b = 1.6$ *and* $a = 0.74$.

defined through the following piecewise linear expression:

$$y_{kl} = f(x_{kl}) = \frac{1}{2}\left(|x_{kl} + 1| - |x_{kl} - 1|\right) \qquad (2)$$

Finally r denotes the neighborhood of interaction of each cell; $\boldsymbol{A}$ and $\boldsymbol{B}$ are linear templates, that are assumed to be space-invariant and I_{kl} is the constant bias term.

The description of the structure is completed by the specification of the boundary conditions, that we assume to be null. For the sake of simplicity, we also assume that the input and the bias terms are null; however we remark that the results presented in the paper also applies, with slight modifications, to the case of constant inputs and non-zero boundary conditions.

3. CNN INSTABILITY AND COMPLEX DYNAMICS

In most CNN applications the input image is loaded either as the initial state x_0 or as a constant input u. The output image is obtained through the time evolution of the network and is extracted from the output voltage y. In order to ensure a proper working of the network it is needed that the CNN be either completely stable (i.e. convergent, according to [13]) or stable almost everywhere (i.e. stable with the exception of a set of initial conditions of measure zero). A rigorous definition of the complete stability and stability almost everywhere is reported below:

Definition 1: a CNN, described by the state equation (1) is said to be completely stable or convergent if for each initial condition, the corresponding trajectory converges towards an equilibrium point.

Definition 2: a CNN, described by the state equation (1) is said to be stable almost everywhere if for each initial condition (with the exception of a set of measure zero), the corresponding trajectory converges towards an equilibrium point.

From a theoretical point of view the difference between the above two kinds of stability is that a CNN stable almost everywhere may present unstable invariant sets, that are not equilibrium points (like for example unstable limit cycles); on the other hand, the only invariant sets of convergent CNNs are either stable or unstable equilibrium points. Such differences, however, cannot be appreciated and observed in real circuits, implementing CNNs. This justifies the introduction of the following definitions:

Definition 3: a CNN is said to be stable if it is either convergent or stable almost everywhere.

Definition 4: a CNN is said to be unstable if it not stable.

According to the above Def. 4, an unstable CNN should present at least one attractor (i.e. a stable invariant set) that is not an equilibrium point. Note that this does not prevent that some attractors still be equilibrium points (see for example the two-cell CNN introduced in [14], that exhibits the coexistence of two stable equilibrium points and one stable limit cycle and that, according to Def. 4 is classified to be unstable). As a consequence of Def. 4, the following proposition holds:

Proposition 5: a sufficient condition for the instability, is that a CNN does not possess stable equilibrium points.

Since the purpose of this manuscript is to investigate the complex dynamics in space-invariant CNNs and complex dynamics can only occur in unstable CNNs, we proceed as follows: 1) we yield some sufficient conditions for CNN instability, ensuring that according to proposition 5 the CNN does not admits stable equilibria; 2) we identify the class of templates satisfying the instability conditions and show that complex behavior can be observed in a wide range of the parameter values; 3) for a case study we investigate in detail the bifurcation processes leading to chaotic attractors.

When considering one-dimensional CNNs, it is possible to prove the following theorem.

Theorem 6: A one-dimensional CNN with N cells, described by a three element template of type $\boldsymbol{A} = [A_{-1}, A_0, A_1]$ has at least one stable equilibrium point if and only if the CNN, composed of only two cells and with the same template, has at least one stable equilibrium point.

As a consequence, for a CNN composed of an arbitrary number of cells, the set of template element values, ensuring the absence of stable equilibrium points, can be easily determined by studying the very simple case of a two-cell CNN. The following result is obtained:

Theorem 7: A one-dimensional CNN with N cells, described by a three element template of type $\boldsymbol{A} = [A_{-1}, A_0, A_1]$ has at least one stable equilibrium point if and only if one of the following two conditions is verified: a) $A_1 A_{-1} > 0$; b) $A_0 - 1 > \min(|A_{-1}|, |A_1|)$.

As far as the $N \times M$ case is concerned, the following theorem holds.

Theorem 8: A two-dimensional CNN composed by $N \times M$ cells, and described by a 3×3 space-invariant template

$$\boldsymbol{A} = \begin{bmatrix} A_{-1,-1} & A_{-1,0} & A_{-1,1} \\ A_{0,-1} & A_{00} & A_{01} \\ A_{1,-1} & A_{10} & A_{11} \end{bmatrix} \tag{3}$$

has at least one stable equilibrium point if and only if the CNN, composed by only 2×2 cells and with the same template, has at least one stable equilibrium point.

Also in this case, the investigation of a simple 2×2 CNN, composed by only two cells, allows to establish conditions for the instability of an arbitrary large CNN. The following result holds:

Theorem 9: A two-dimensional CNN with $N \times M$ cells, described by a 3×3 template $\boldsymbol{A}$ admits of at least one stable equilibrium point if and only if there exist $p_1, p_2, p_3 \in \{-1, 1\}$ such that the following set of inequalities is satisfied:

$$\begin{aligned} A_{00} - 1 + A_{01}\, p_1 + A_{10}\, p_2 + A_{11}\, p_3 &> 0 \\ A_{00} - 1 + A_{0,-1}\, p_1 + A_{1,-1}\, p_1 p_2 + A_{10}\, p_1 p_3 &> 0 \\ A_{00} - 1 + A_{-1,0}\, p_2 + A_{-1,1}\, p_1 p_2 + A_{01}\, p_2 p_3 &> 0 \\ A_{00} - 1 + A_{-1,0}\, p_1 p_3 + A_{-1,-1}\, p_3 + A_{0,-1}\, p_2 p_3 &> 0 \end{aligned}$$

The time-domain simulation have shown that for most of the unstable templates satisfying either Theorem 7 or 9, the most common behavior is chaotic, even in networks composed by a limited number of cells.

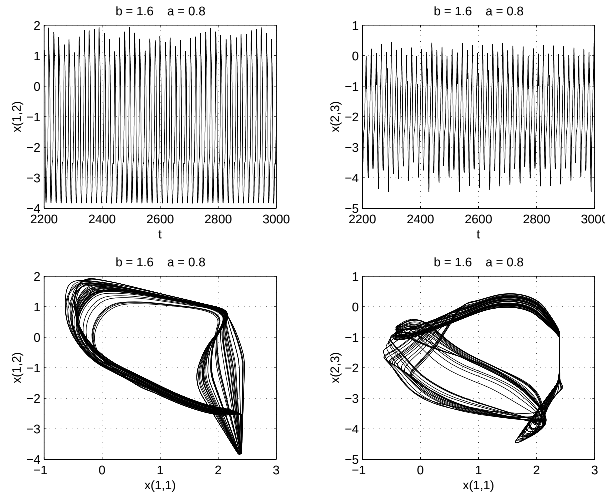

Figure 3: *Chaotic attractor in a CNN composed by 9 cells and described by the template (4) with* $b = 1.6$ *and* $a = 0.8$.

As a case study we have considered a CNN composed by 3×3 cells and described by the following template

$$\boldsymbol{A} = \begin{bmatrix} a & a & a \\ -a & b & a \\ -a & a & -a \end{bmatrix} \quad \text{with} \;\; b = 1.6; \;\; a > 0.6 \tag{4}$$

The results of the simulations are shown in Figs. 1-5; we have reported the steady-state waveforms of cells x_{12} and x_{23} and the projection of the steady-state trajectory onto the two planes (x_{11}, x_{12}) and (x_{11}, x_{23}).

The following phenomena are observed: for $a > 0.6$ (see Fig. 1) the CNN does not possess any more equilibrium points and a limit cycle occurs (probably through an heteroclinic bifurcation as described in [6] for a smaller network); by increasing a a typical period doubling bifurcation is observed (Fig. 2), leading to the chaotic attractor shown in Fig. 3. By further increasing a several periodic windows are encountered (see Fig. 4) separating different chaotic regions (see for example the chaotic attractor observed for $a = 0.94$ in Fig. 5).

Extensive simulations, that for lack of space are not reported here, have revealed the existence of a wide gallery of chaotic attractors and of several period doubling bifurcation phenomena. The existence of these bifurcation processes has also been checked through the computation of the limit cycle Floquet's multipliers.

4. CONCLUSIONS

We have investigated the occurrence of complex dynamic behaviors (i.e. bifurcation processes, strange and chaotic attractors) in autonomous space-invariant CNNs. There are

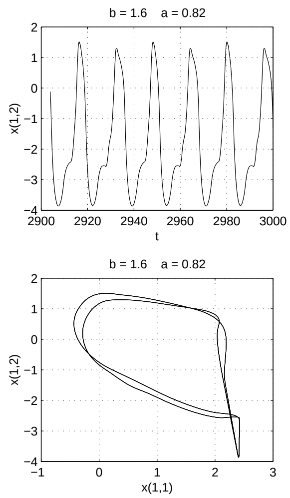

Figure 4: *Periodic window in a CNN composed by 9 cells and described by the template (4) with* $b = 1.6$ *and* $a = 0.82$.

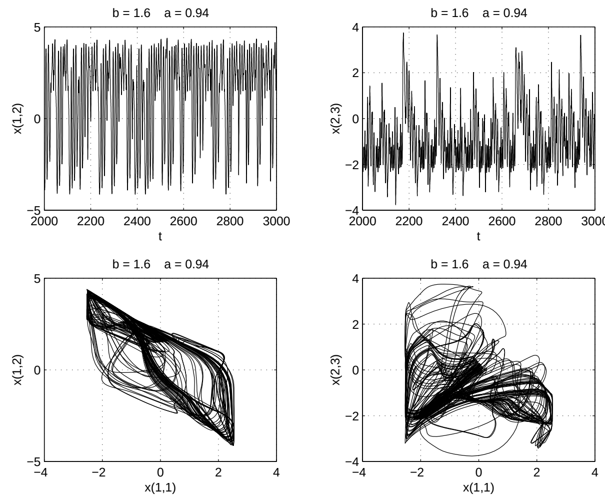

Figure 5: *Chaotic attractor in a CNN composed by 9 cells and described by the template (4) with* $b = 1.6$ *and* $a = 0.94$.

some reasons for carrying on this study: a) most CNN implementations exploit space-invariant templates; b) so far no example of complex dynamics has been shown in autonomous space-invariant CNNs. Hence, we are confident that the identification of chaotic dynamics in such networks might open the possibility of developing, on the existing CNN chips, innovative chaos-based applications.

The principal results presented in the paper are the following: a) we have provided some sufficient conditions for the instability of a space-invariant CNN (i.e. ensuring that the network does not present stable equilibrium points): such conditions can be easily checked by looking at the template elements; b) we have verified that unstable CNNs, besides the occurrence of limit cycles, quite often exhibit bifurcation phenomena, leading to strange and chaotic attractors; d) for a case-study we have investigated the bifurcation processes leading to chaos.

5. REFERENCES

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