On the Dynamic Behavior of Cellular Neural Networks

M. Gilli, F. Corinto, M. Biey and P.P. Civalleri
Dipartimento di Elettronica, Politecnico di Torino
Corso Duca degli Abruzzi, 24
I-10129 Torino, Italy

Abstract - Cellular neural networks (CNNs) are analog dynamic processors that have found several applications for the solution of complex computational problems. The mathematical model of a CNN consists in a large set of coupled nonlinear differential equations that have been mainly studied through numerical simulations; the knowledge of the dynamic behavior is essential for developing rigorous design methods and for establishing new applications. CNNs can be divided in two classes: stable CNNs, with the property that each trajectory (with the exception of a set of measure zero) converges towards an equilibrium point; unstable CNNs with either a periodic or a non/periodic (possibly complex) behavior. The manuscript is devoted to the comparison of the dynamic behavior of two CNN models: the original Chua-Yang model and the Full Range model, that was exploited for VLSI implementations.

I. Introduction

Cellular neural networks (CNNs) are analog dynamic processors, suitable for solving all the computational problems that can be formulated in terms of local interactions among signals placed on a regular structure [1], [2], [3]. The fundamental property that distinguishes a CNN from a general neural network is the local connectivity, that has allowed the realization of several high-speed complex VLSI chips.

The original model of a CNN cell was introduced by Chua and Yang in [1]. As far as the dynamics is concerned, the Chua-Yang CNNs (CYCNNs) can be classified in two categories: stable CNNs, unstable CNNs with a periodic behavior or a non/periodic (possibly complex) behavior. In particular, CYCNN stability was widely investigated: the main results are reviewed and summarized in [4]. For what concerns unstable CYCNNs (i.e CNNs which exhibit at least one attractor that is not a stable equilibrium point) some examples of periodic limit cycles were shown in [5]-[8]; some examples of complex (chaotic) behavior were recently reported in [9].

The hardware realization of large-complexity CNN chips has required to modify to original Chua-Yang model, into a new one, called Full Range model [10], [11]. Such a new model presents the characteristic of reducing the signal-range of the state variables. Practical experiments, developed on the existing VLSI chips, and extensive computer simulations have shown that in most cases the qualitative dynamics of Full Range CNNs (FRCNNs) is very similar to that of CYCNNs. Such experiments, however, were mainly related to stable networks, exploited for image-processing applications. In addition, from a mathematical point of view, the two CNN models are described by rather different systems of differential equations: as to the author knowledge, no complete study about FRCNN dynamics has been, so far, presented.

The manuscript is organized in four sections. In the first section the CYCNN and the FRCNN models are rigorously defined, by assuming that the output functions of both the models admit of a piecewise linear approximation. In the second section we will briefly summarize the most significant results, regarding the stability of CYCNNs; such results are extended to FRCNNs, where it is possible, and the stability properties of the two models are compared. In the third section we will investigate the periodic and non-periodic behavior of both CYCNNs and FRCNNs. Finally the fourth section is devoted to the Conclusions.

II. The Chua-Yang and the Full Range Models

We consider CNNs composed by $N \times M$ cells arranged on a regular grid and denote the position of a cell with two indexes (k, l).

A CYCNN is governed by the following normalized state equations

$$\begin{aligned} \dot{x}_{kl} &= -x_{kl} + \sum_{|n|\le r, |m|\le r} \hat{A}_{nm}\, y_{k+n,l+m} \\ &\quad + \sum_{|n|\le r, |m|\le r} \hat{B}_{nm}\, u_{k+n,l+m} + I \end{aligned} \qquad (1)$$

0-7803-7278-6/02/$10.00

where x_{kl} and u_{kl} represent the state-voltage and the input voltage of cell (k,l); y_{kl} is the output voltage, defined through the following piecewise linear expression:

$$y_{kl} = f(x_{kl}) = \frac{1}{2}\left(|x_{kl}+1| - |x_{kl}-1|\right) \tag{2}$$

Finally r denotes the neighborhood of interaction of each cell; $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ are linear templates, that are assumed to be space-invariant and I is the bias term.

An alternative and useful expression for the state equation of a CYCNN is obtained by ordering the cells in some way (e.g. by rows or by columns) and by repacking the state, the input, the output variables and the bias terms into the vectors $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{y}$, $\boldsymbol{I}$. The following compact form is obtained:

$$\dot{\boldsymbol{x}} = -\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{I} \tag{3}$$

where matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are obtained through the templates $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$, as explained in [12].

Full Range CNNs, that have shown to be suitable for VLSI implementation, are described by the following mathematical model

$$\begin{aligned} \dot{z}_{kl} \quad &= \quad -z_{kl} + \sum_{|n|\le r,|m|\le r} \hat{A}_{nm}\, z_{k+n,l+m} - g(z_{kl}) \\ &\quad + \sum_{|n|\le r,|m|\le r} \hat{B}_{nm}\, u_{k+n,l+m} + I \end{aligned} \tag{4}$$

where the state is now represented by z_{kl}, whereas u_{kl}, I, $\hat{A}_{nm}$ and $\hat{B}_{nm}$ still represent the input, the bias and the linear templates defined above. For the sake of the simplicity, we assume that the function $g(\cdot)$ admits of the following piecewise linear expression:

$$g(z_{ij}) = \begin{cases} h(z_{ij}+1) & z_{ij} < -1 \\ 0 & |z_{ij}| \le 1 \\ h(z_{ij}-1) & z_{ij} > 1 \end{cases} \tag{5}$$

where h is supposed to be large enough for approximating the nonlinear characteristic shown in Fig. 4 of [11].

By ordering the state, the input, and the bias terms into the vectors $\boldsymbol{z}$, $\boldsymbol{u}$ and $\boldsymbol{I}$, the following compact form is obtained:

$$\dot{\boldsymbol{z}} = -\boldsymbol{z} + \boldsymbol{A}\boldsymbol{z} - g(\boldsymbol{z}) + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{I} \tag{6}$$

We remark that, due to the piecewise approximation of the output functions $f(\cdot)$ and $g(\cdot)$, the state space of both a CYCNN and a FRCNN is composed by regions, that can be classified as follows: a) linear regions if all the cells lye in the linear part of their characteristic (i.e. $|z_{ij}| \le 1$ and $|x_{ij}| \le 1$); b) saturation regions if all the cells are saturated (i.e $|z_{ij}| > 1$ and $|x_{ij}| > 1$); c) partial saturation regions if some cells are saturated and some others are not.

A CNN composed by $N \times M$ cells exhibits 3^{NM} regions.

III. Stability of the two CNN models

In image processing applications the input image is loaded either as the initial state $\boldsymbol{x}_0$ or as a constant input $\boldsymbol{u}$. The output image is obtained through the time evolution of the network and is extracted from the output voltage $\boldsymbol{y}$. In order to ensure a proper working of the network it is needed that the CNN be completely stable or stable almost everywhere (i.e. stable with the exception of a set of initial conditions of measure zero). The rigorous definition of these two type of stability is reported below:

Definition 1: An autonomous dynamical system, described by the state equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in R^n, \quad \boldsymbol{f} : R^n \to R^n \tag{7}$$

is said to be completely stable (or convergent) if for each initial condition $\boldsymbol{x}_0 \in R^n$

$$\lim_{t\to\infty} \boldsymbol{x}(t, \boldsymbol{x}_0) = \text{const} \tag{8}$$

where $\boldsymbol{x}(t, \boldsymbol{x}_0)$ is the trajectory starting from $\boldsymbol{x}_0$.

Definition 2: An autonomous dynamical system, described by the state equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in R^n, \quad \boldsymbol{f} : R^n \to R^n \tag{9}$$

is said to be stable (or convergent) almost everywhere if for each initial condition $\boldsymbol{x}_0 \in R^n$ (with at most the exception of a set of measure zero)

$$\lim_{t\to\infty} \boldsymbol{x}(t, \boldsymbol{x}_0) = \text{const} \tag{10}$$

where $\boldsymbol{x}(t, \boldsymbol{x}_0)$ is the trajectory starting from $\boldsymbol{x}_0$.

The complete stability of the Chua-Yang model was studied in several papers [13], [14], [15], [16] including the original paper, where the CNN paradigm was introduced [1]. The results are summarized in the following Theorems [4]:

Theorem 1: A sufficient condition for the complete stability of a CYCNN, described by state equations (3), is that there exists a positive diagonal matrix $\boldsymbol{D}$, such that the product $\boldsymbol{D}\boldsymbol{A}$ is a symmetric matrix.

Definition 3: A matrix P with positive diagonal elements is said to be *strictly diagonal dominant* if:

$$\forall\, i, \quad p_{ii} > \sum_{j\ne i} |p_{ij}| \tag{11}$$

Theorem 2: A sufficient condition for the complete stability of a CYCNN, described by state equations (3), is that the comparison matrix of $\boldsymbol{A} - \boldsymbol{I}$ is a nonsingular M-matrix (or equivalently that there exists a positive diagonal matrix $\boldsymbol{D}$ such that $(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{D}$ is strictly diagonally dominant).

Stability almost everywhere was investigated in [17]: the results are based on the mathematical analysis presented in [18] for cooperative systems of differential equations. The following Theorem holds [4]:

Theorem 3: A sufficient condition for the stability almost everywhere of a CYCNN, described by state equations (3), is that the matrix $\boldsymbol{A}$ is irreducible and there exists a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{DAD}^{-1}$ presents non-negative off-diagonal elements.

The stability investigation of the Full Range model allows to prove very similar properties. The following theorems can be proved:

Theorem 4: A sufficient condition for the complete stability of a FRCNN, described by (6), is that there exists a positive diagonal matrix $\boldsymbol{D}$, such that the product $\boldsymbol{DA}$ is a symmetric matrix and that all the equilibrium points are isolated.

Proof: We assume that the input $\boldsymbol{u}$ and the bias term $\boldsymbol{I}$ be null: the extension of the proof to nonzero inputs and bias simply requires slight modifications. Let $\boldsymbol{T}$ be a diagonal matrix defined as $\boldsymbol{T} = \text{diag}(t_1, ..., t_n)$ such that $\boldsymbol{D} = \boldsymbol{T}^2$. Let us consider the state transformation $\boldsymbol{w} = \boldsymbol{Tz}$. The state equation (6) can be written as:

$$\dot{\boldsymbol{w}} = \boldsymbol{T}\dot{\boldsymbol{z}} = -\boldsymbol{w} + \boldsymbol{TAT}^{-1}\boldsymbol{w} - \boldsymbol{T}g(\boldsymbol{T}^{-1}\boldsymbol{w}) \qquad (12)$$

We denote with $\boldsymbol{P}$ the matrix $\boldsymbol{P} = \boldsymbol{TAT}^{-1} - \boldsymbol{U}$, where $\boldsymbol{U}$ denotes the identity matrix. Due to the hypotheses of the Theorem $\boldsymbol{DA} = \boldsymbol{T}^2\boldsymbol{A}$ is symmetric; hence the matrix $\boldsymbol{P} = \boldsymbol{T}^{-1}\boldsymbol{DAT}^{-1} - \boldsymbol{U}$ is also symmetric. The state equation (12) admits of the following form:

$$\dot{\boldsymbol{w}} = \boldsymbol{Pw} - \boldsymbol{T}g(\boldsymbol{T}^{-1}\boldsymbol{h}) \qquad (13)$$

Let us consider the Lyapunov function:

$$V(\boldsymbol{w}(t)) = -\frac{1}{2}\boldsymbol{w}^\top \boldsymbol{Pw} + \sum_i \int_{t_i^{-1}w_i(0)}^{t_i^{-1}w_i} t_i^2 g_i(v)dv. \qquad (14)$$

Since P is symmetric, the time derivative of $V(t)$ yields:

$$\begin{aligned} \dot{V}(t) &= -\dot{\boldsymbol{w}}^\top \boldsymbol{Pw} + \sum_i t_i^2 g_i(t_i^{-1}w_i)t_i^{-1}\dot{w}_i \\ &= -\dot{\boldsymbol{w}}^\top[\boldsymbol{Pw} - \boldsymbol{T}g(\boldsymbol{T}^{-1}\boldsymbol{w})] \\ &= -\dot{\boldsymbol{w}}^\top \dot{\boldsymbol{w}} \le 0 \qquad (15) \end{aligned}$$

Owing to the fact that all the trajectories are bounded, by La Salle's invariance principle they approach the set $M = \{\boldsymbol{h} : V(\boldsymbol{h}) = 0\}$ which coincides exactly with the set of equilibrium points of the dynamical system (15). If such equilibrium points are isolated, La Salle's principle implies that the CNN is completely stable, according to Definition 1.

Theorem 5: A sufficient condition for the complete stability of a FRCNN, described by (6), is that the comparison matrix of $\boldsymbol{A} - \boldsymbol{I}$ is a nonsingular M-matrix (or equivalently that there exists a positive diagonal matrix $\boldsymbol{D}$ such that $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{D}$ is strictly diagonally dominant).

Proof: If $\boldsymbol{D}$ is the identity matrix, the Theorem is easily proved by use of the following arguments : a) due to the diagonally dominance property, each cell that reaches a saturated value ± 1 cannot enter back the linear part of its characteristic; in addition the linear region and the partial saturated regions should exhibit at least one eigenvalue with positive real part; b) due to a) a steady-state trajectory cannot belong to two or more regions of the state space; moreover periodic and nonperiodic orbits belonging to a single region are prevented. It is derived that each trajectory should converge to an equilibrium point, i.e. the network is completely stable. If $\boldsymbol{D}$ is a generic positive diagonal matrix, the same arguments can be applied by considering the state transformation $\boldsymbol{w} = \boldsymbol{D}z$.

Theorem 6: A sufficient condition for the stability almost everywhere of a FRCNN, described by state equations (6), is that the matrix $\boldsymbol{A}$ is irreducible and there exists a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{DAD}^{-1}$ presents non-negative off-diagonal elements.

Proof: Under the state transformation $\boldsymbol{w} = \boldsymbol{D}z$ the state equation (6) can be written as:

$$\dot{\boldsymbol{w}} = \boldsymbol{D}\dot{\boldsymbol{z}} = -\boldsymbol{w} + \boldsymbol{DAD}^{-1}\boldsymbol{w} - \boldsymbol{D}g(\boldsymbol{D}^{-1}\boldsymbol{w}) \qquad (16)$$

If $\boldsymbol{DAD}^{-1}$ exhibits non-negative off-diagonal elements, then the system (16) is cooperative. Hirsch's results for cooperative systems [18] ensures the stability almost everywhere.

The above results shows that the conditions under which stability can be proved are almost identic for both CYCNNs and FRCNNs. In addition to that, it was shown in [10] that there is a one to one correspondence between the equilibrium points of the CYCNN described by (3) and the FRCNN described by (6): such a correspondence holds also for the stability characteristics (eigenvalues of the Jacobian matrix) of each equilibrium point. This allows to conjecture that also the dynamics of the two CNN models be similar.

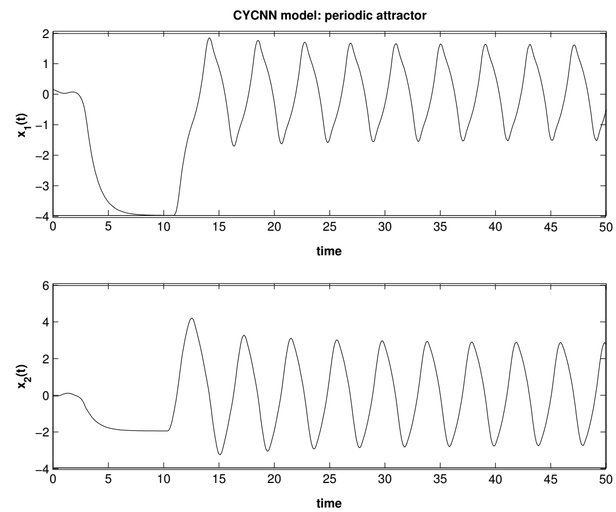

Fig. 1. *Periodic waveforms in a CYCNN, described by an opposite-sign template*

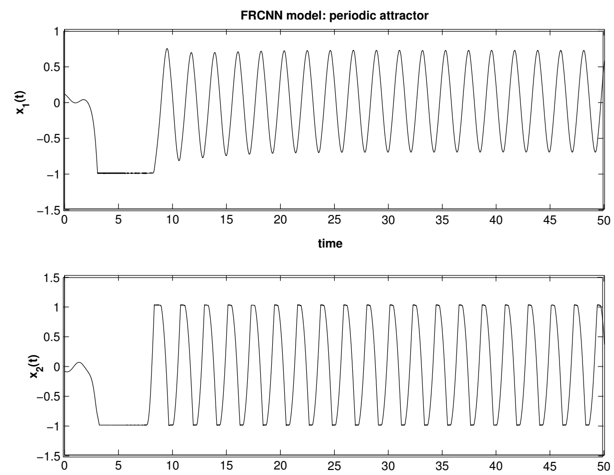

Fig. 2. *Periodic waveforms in a FRCNN, described by an opposite-sign template*

IV. Periodic and non-periodic behavior of the two CNN models

The investigation of the dynamic behavior of unstable CNNs, i.e. networks that exhibit either a periodic or a non periodic behavior is more complex than the study of stability. Apart from some studies, based on spectral techniques [19], [20], [21], most of the results are based on numerical simulations.

We report here two statements, that compare the global dynamic behavior of a CYCNN and a FRCNN. The proofs are already in progress: however the statements will be reported as conjectures, because some delicate technical aspects of the proofs have not been completed so far.

Conjecture 1: Let us assume that the CYCNN, described by (3) presents a steady state periodic limit cycle, crossing a sequence of region R_1, $...R_N$, R_1. Then the FRCNN, described by equation (6) also exhibits a periodic limit cycle, that crosses the same sequence of regions R_1, $...R_N$, R_1

Conjecture 2: Let us assume that the CYCNN, described by (3) presents a generic trajectory crossing a sequence of region $R_1, ...R_N, ...$ Then the FRCNN, described by equation (6) also exhibits a trajectory, that crosses the same sequence of regions $R_1, ...R_N, ...$

The validity of the above conjectures have also been verified through numerical simulations. As an example, we report here the simulation of a one-dimensional network composed by 12 cells and described by the opposite-sign template $[-3, 2, 3]$. Figs. 1 and 2 shows the time waveforms of the first and the second cell for the Chua-Yang and the Full Range model respectively: it is seen that both the CYCNN and the FRCNN model exhibit a periodic limit cycle. We also mention the results reported in ([9]), where it is shown that both a CYCNN and a FRCNN model may exhibit a complex dynamic behavior for the template below (with $b = 1.6$ and $a = 1$):

$$\hat{\boldsymbol{A}} = \begin{bmatrix} a & a & a \\ -a & b & a \\ -a & a & -a \end{bmatrix} \quad \text{with } b = 1.6; \; a = 1 \qquad (17)$$

The non-periodic attractors (possibly chaotic, see [9]) presented by the two CNN models are reported in Fig. 3 and 4.

Additional significant examples will be reported in the final version of the manuscript.

V. Conclusions

Cellular Neural Networks are complex dynamical systems described by a large set of nonlinear differential equations. The knowledge of their dynamic behavior is essential in order to develop rigorous design methods and new applications.

Two CNN models have been considered in the literature: the original Chua-Yang model, that was introduced in the seminal paper of L.O. Chua [1], [2] and represents the starting point of all the CNN studies and applications; the Full Range model, that was introduced in [10], [11] and is more suitable for VLSI implementation.

In this manuscript we have compared the dynamic behavior of the two models for what concerns the stability properties and the global dynamic behavior. We have

0-7803-7278-6/02/$10.00 ©2002 IEEE

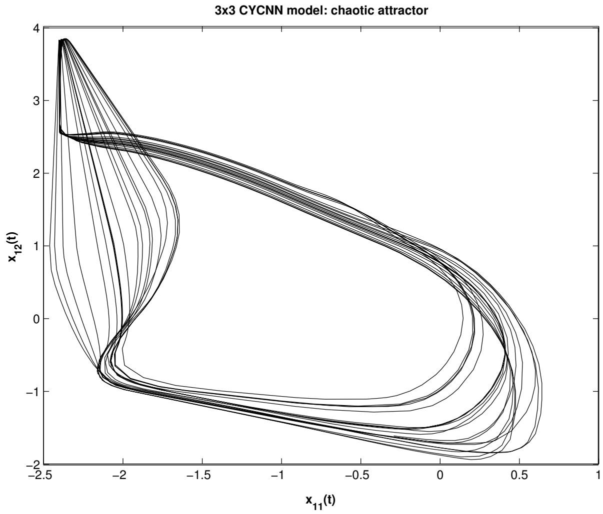

Fig. 3. *Non-periodic attractor in a 3×3 CNN described by template (17).*

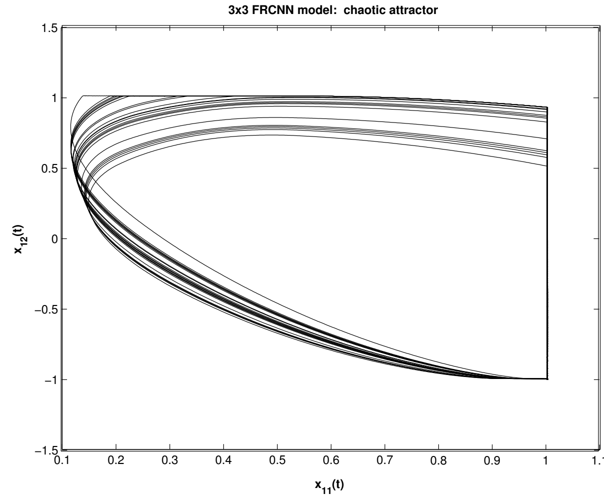

Fig. 4. *Non-periodic attractor in a 3×3 CNN described by template (17).*

proved that the conditions under which complete stability (or stability almost everywhere) can be proved are almost identic for the two models. Then we have presented two statements, regarding the global dynamic behavior of the two models: they claim that there is a one to one correspondence between the periodic and the nonperiodic invariant limit sets of the two CNN models. The statements are enunciated as conjectures; their proofs are already in progress, but some technical details are still missing. The conjectures have been verified through several numerical simulations.

VI. Acknowledgments

This work was supported in part by *Ministero dell'Istruzione, dell'Università e della Ricerca*, under the National Research Plan PRIN 2001 and by *Consiglio Nazionale delle Ricerche (CNR)*, under contract no. CNRG000D51.

References

[1] L.O.Chua and L.Yang,"Cellular Neural Networks: Theory" *IEEE, Trans. Circuits and Syst.*, vol. 35, no. 10, pp. 1257-1272, Oct. 1988.

[2] L.O.Chua and L.Yang, "Cellular Neural Networks: Applications," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1273-1290, Oct. 1988.

[3] L.O.Chua and T. Roska, "The CNN Paradigm," *IEEE, Trans. Circuits and Syst.-I*, vol. 40, no. 3, pp. 147-156, March 1993.

[4] P. P. Civalleri, M. Gilli, "On Stability of Cellular Neural Networks," *Journal of VLSI Signal Processing*, Kluwer Academic Publisher, vol. 23, no. 2/3, pp. 429-435, 1999.

[5] F.Zou, J.A.Nossek, "Stability of Cellular Neural Networks with Opposite-Sign Templates," *IEEE, Trans. Circuits and Syst.*, vol. 38, no. 6, pp. 675-677, June 1991.

[6] F.Zou, J.A.Nossek, "A Chaotic Attractor with Cellular Neural Networks", Opposite -Sign Templates," *IEEE, Trans. Circuits and Syst.-I*, vol. 38, no. 7, pp. 811-812, July 1991.

[7] F.Zou, J.A.Nossek, "Bifurcation and Chaos in CNN's," *IEEE, Trans. Circuits and Syst.-I*, vol. 40, no. 3, pp. 166-173, March 1993.

[8] P.P. Civalleri, and M. Gilli, "Global Dynamic Behaviour of a Three Cell Connected Component Detector CNN," *Int. J. Circuit Theory and Applicat.*, vol. 23, pp. 117-135, March-April 1995.

[9] M. Biey, M. Gilli, and P. Checco, "Complex Dynamic Phenomena in Space-Invariant Cellular Neural Networks," to appear on *IEEE, Trans. Circuits and Syst.-I*, vol. 49, March 2002.

[10] S. Espejo, R. Carmona, R. Dominguez-Castro, and A. Rodriguez Vàzquez, "A VLSI Oriented Continuous TIme CNN Model," *Int. J. Circuit Theory and Applicat.*, vol. 24, pp. 341-356, 1996.

[11] G. Linan, R. Domínguez-Castro, S. Espejo, and A. Rodríguez-Vázquez, "Design of Large-Complexity Analog I/O CNNUC," *ECCTD'99 - Design Automation Day on Cellular Visual Microprocessor*, pp. 42-57, Stresa (Italy), 1999.

[12] L.O.Chua and T.Roska, "Cellular Neural Networks with Nonlinear and Delay-Type Template Elements and Non-UnIform Grids," *International Journal on Circuit theory and Applications*, vol. 20, pp. 449-451, 1992.

[13] C.W.Wu and L.O.Chua, "A More Rigorous Proof of Complete Stability of Cellular Neural Networks," *IEEE Transactions on Circuits and Systems: I*, vol. 44, no. 4, pp. 370-371, April 1997.

[14] M. Gilli, "Stability of Cellular Neural Networks and Delayed Cellular Neural Networks with Nonpositive Templates and Non-Monotonic Output Functions," *IEEE Transactions on Circuits and Systems - I*, vol. 41, no. 8, pp. 518-528, Aug. 1994.

[15] S. Arik, V. Tavsanoglu, "Equilibrium analysis of non-symmetric CNNs," *International Journal on Circuit theory and Applications*, vol. 24, pp. 269-674, 1996.

[16] N. Takahashi and L. O. Chua, "On the Complete Stability of Nonsymmetric Cellular Neural Networks," *IEEE Transactions*

on Circuits and Systems - I, vol. 45, no. 7, pp. 754-758, July 1998.

[17] L.O.Chua and C.W.Wu, "The Universe of Stable CNN Templates," *International Journal of Circuit Theory and Applications: Special Issue on Cellular Neural Networks*, Vol. 20, pp. 497-517, July-August 1992.

[18] M.W.Hirsch, "System of Differential Equations that are Competitive and Cooperative II: Convergence Almost Everywhere", *SIAM J. Math. Anal.*, Vol. 16, pp. 423-439, 1985.

[19] G. Setti, P. Thiran, and C. Serpico, "An Approach to Information Processing in 1-dimensional Cellular Neural Networks - Part II: Global propagation," *IEEE, Trans. Circuits and Syst.-I*, vol. 45, July 1998.

[20] P.P. Civalleri and M. Gilli, "A spectral Approach to the Study of Propagation Phenomena in CNNs," *Int. J. Circuit Theory and Applicat.*, vol. 24, pp. 37-47, January-February 1996.

[21] M. Gilli, "Analysis of Periodic Oscillations in Finite-Dimensional CNNs Through a Spatio-Temporal Harmonic Balance Technique", *Int. J. Circuit Theory and Applicat.*, vol. 25, pp. 279-288, July-August 1997.