

Influence of Topology on Synchronization in Networks of Coupled Hindmarsh-Rose Neurons

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Abstract—Synchronization plays a central role in information processing in many systems. In this work, starting from a method for predicting the behavior of the synchronous state in a network of Hindmarsh-Rose neurons, the dependence of the synchronization properties of the network on the topology is shown.

I. INTRODUCTION

During the last few years networks of bio-inspired neurons have interested an increasing number of researchers in all branches of science. In particular, spiking neurons have attracted the interest because many studies consider this behavior an essential component in information processing by the brain [1]. In this class of neurons, bursting neurons are of relevant interest because they characterize a variety of biological oscillators. The electrical potential of these neurons, which typically is the state variable that contains the main information, undergoes a succession of alternating active and silent phases in which, respectively, it has a spiking behavior (very fast oscillations) and then evolves slowly without oscillations. Furthermore, many studies suggest that retrieval of the stored patterns is related to spontaneously occurring synchrony in networks of neurons (see [2].) This motivates the investigation of the conditions for synchronization in networks of bursting neurons [3]–[6].

The synchronization conditions of a network of Hindmarsh-Rose neurons have been studied in several papers (for example see [3], [4]) and more detailed conditions have been recently introduced in [5], where two different types of coupling have been considered. In the case of synaptic coupling the synchronous behavior may be different from the behavior of an isolated neuron.

In [6] the bifurcation diagram of the synchronous state with respect to the product between the coupling strength g_s and the number k of connections of each cell (called *degree* hereafter) has been introduced. Therefore, a complete method to analyze and synthesize the synchronous state of a network of Hindmarsh-Rose neurons by combining the synchronization conditions of the network, obtained in [5], and the bifurcation diagram of the synchronous state, proposed in [6], is available. Thanks to this method, it is possible to verify that the synchronization properties depend on the topology of the network: this is the main aim of this paper.

At first, in Section II some preliminaries are introduced: the Hindmarsh-Rose neuron model, the coupling function, the network equations, and the synchronous state equations. Then, in Section III, the synchronization properties presented in [5] and the bifurcation diagram presented in [6] are summarized. In Section IV the influence of topology on the synchronization properties is shown. Conclusions (Section V) close the paper.

II. PRELIMINARIES

The Hindmarsh-Rose neuron model [7], a simplified version of the Hodgkin-Huxley model and a modification of the FitzHugh equations – originally proposed to model the synchronization of firing of two snail neurons – can be described by the following equations [4], [8]:

$$\begin{cases} \dot{x}(t) &= f_x(x, y, x) &= ax^2 - x^3 - y - z \\ \dot{y}(t) &= f_y(x, y, x) &= (a + \alpha)x^2 - y \\ \dot{z}(t) &= f_z(x, y, x) &= \mu(bx + c - z). \end{cases} \quad (1)$$

In these equations $x(t)$ represents the membrane potential, usually considered as a natural output of the cell, and $y(t)$ and $z(t)$ are the recovery and the adaptation variables, which account for fast and slow ion currents respectively. Let us use the same parameters as in [4], [5], namely $a = 2.8$, $b = 9$, $c = 5$, $\alpha = 1.6$, and $\mu = 0.001$. In Fig. 1 the time evolution of the state variables is shown, where it is possible to see that, after a short transient, it is periodic.

It is possible to model the coupling in a network of N such neurons in different ways [5], but in this work we focus on synaptic coupling between the x variables. The synaptic coupling can be modeled as a (static nonlinear) sigmoidal function, which is the simplest one in a neuronal system [9]:

$$\gamma(x_j) = \frac{1}{1 + e^{-\nu(x_j - \theta_s)}} \quad (2)$$

where the two free parameters are chosen to be $\nu = 10$ and $\theta_s = -0.25$. The evolution of the i -th neuron is then ruled by

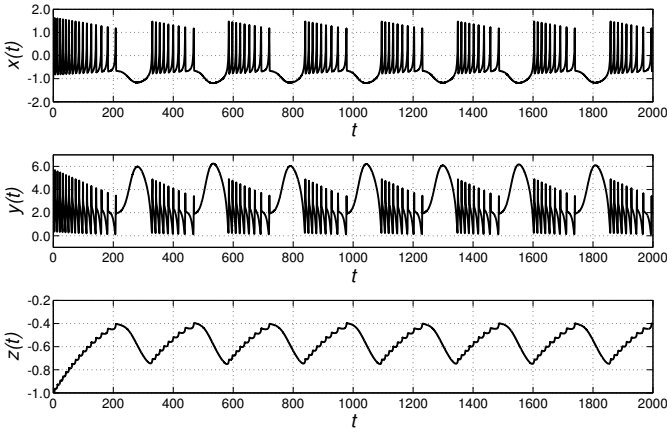


Fig. 1. Time evolution of the state variables of an isolated neuron according to the Hindmarsh-Rose model.

the following equations

$$\begin{cases} \dot{x}_i(t) = f_x(x_i, y_i, x_i) - g_s(x_i - V_s) \sum_{j=1}^N c_{ij} \gamma(x_j) \\ \dot{y}_i(t) = f_y(x_i, y_i, x_i) \\ \dot{z}_i(t) = f_z(x_i, y_i, x_i) \end{cases} \quad (3)$$

where g_s is the coupling strength, V_s is the reversal potential, assumed to be $V_s = 2$, and c_{ij} are the elements of the *adjacency matrix* \mathbf{C} : $c_{ii} = 0$, $c_{ij} = c_{ji} = 1$ if neurons i and j are connected to each other, and $c_{ij} = c_{ji} = 0$ otherwise.

Let $\xi_i = (x_i, y_i, z_i)'$, $\mathbf{f} = (f_x, f_y, f_z)'$, and $\Gamma(\xi_j) = (\gamma(x_j), 0, 0)'$, Equation (3) can be recast as follow:

$$\dot{\xi}_i = \mathbf{f}(\xi_i) - g_s(x_i - V_s) \sum_{j=1}^N c_{ij} \Gamma(\xi_j) \quad (4)$$

In order to obtain the conditions of identical synchronization, the *master stability function* approach (see [10], [11]) has been used in [5]. It has been shown that, by considering the identical synchronization conditions, *i.e.* $\xi_i = \xi^* = (x^*, y^*, z^*)' \forall i$, the identically synchronous state exists only if the sum of c_{ij} is constant with respect i :

$$\sum_{j=1}^N c_{ij} = \text{const} = k. \quad (5)$$

Then, the evolution of the synchronous state is described by the following system of ordinary differential equations:

$$\dot{\xi}^*(t) = \mathbf{f}(\xi^*) - \eta(x^* - V_s)\Gamma(\xi^*) \quad (6)$$

where $\eta = kg_s$ is a parameter of the network taking account of the number k of connections of each node and the coupling strength g_s . In the next Section the synchronization properties of networks of Hindmarsh-Rose neurons, studied in [5], and the bifurcation diagram of the synchronous state, presented in [6], are summarized.

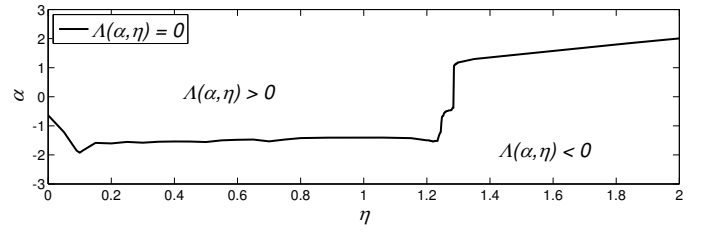


Fig. 2. Zero-Lyapunov exponent curve (with respect η parameter) of the master stability function associated to the synchronous state equation of a Hindmarsh-Rose neuron network.

III. SYNCHRONIZATION PROPERTIES

In Fig. 2 a zero-Lyapunov exponent curve, with respect to η parameter, is reported; this curve is obtained from a modified master stability function associated to the synchronous state equation of a Hindmarsh-Rose network with synaptic coupling (Equation 6). This curve contains all the information about the synchronization. In fact, taken a network, its η parameter is known: $\bar{\eta} = kg_s$. The value of α that correspond to $\bar{\eta}$, according to the zero-Lyapunov exponent curve $\Lambda(\alpha, \eta) = 0$, can be estimated: $\bar{\alpha} = \alpha(\bar{\eta})$. The network is a *Class-A* network (see [5], [11] for details), therefore it synchronizes if and only if the second largest eigenvalue of the adjacency matrix, namely $\lambda_2(\mathbf{C})$, satisfies

$$g_s \cdot \lambda_2(\mathbf{C}) < \bar{\alpha} \quad (7)$$

Thanks to the above results, the synchronization of a network of Hindmarsh-Rose neurons can be verified, but there is an open question about this network yet. In fact, analyzing Equation (6), the synchronous behavior may be different from the behavior of an isolated Hindmarsh-Rose neuron, unless in the trivial case $\eta = 0$. It follows that the behavior of the synchronous state with respect to η parameter has to be analyzed. In Fig. 3 a simplified version of the bifurcation diagram (introduced in [6]) is reported in which only the attractors are shown.

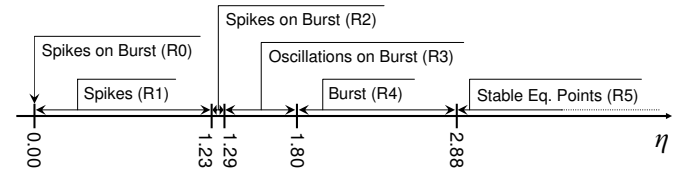


Fig. 3. Simplified bifurcation diagram, reporting only the attractors, of the synchronous state equation with respect to η parameter.

Taking η as bifurcation parameter, it is possible to identify 5 different types of attractors in 5 not overlapping intervals and one isolated point. In the isolated point R0 ($\eta = 0$) and R2 region ($\eta \in (1.23, 1.29)$) a spiking burst behavior is exhibited, *i.e.* there is a succession of two alternating phases (bursts) and the spikes are present only in the active one. Note that the point R0 corresponds to an isolated Hindmarsh-Rose neuron. In R1 region ($\eta \in (0.00, 1.23)$) a spike behavior takes place. On the other hand, the synchronous behavior is composed of damped oscillations on bursts in region R3 ($\eta \in (1.29, 1.80)$).

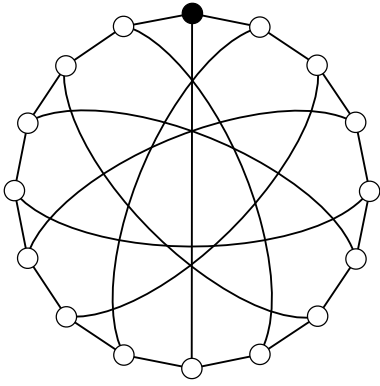


Fig. 4. Topology of a network belonging the class $\mathfrak{N}(16, 3)$. The cells are numbered clockwise starting from the black one.

The synchronous state exhibits a periodic behavior (only burst phases) in R4 region ($\eta \in (1.80, 2.88)$). Finally, in R5 region the behavior reduces to a stable equilibrium point.

Starting from the results summarized in this Section, the synchronization properties can be connected with the topology of the network taking into account that the eigenvalues (spectrum) of the adjacency matrix can change with the topology. This is the aim of the next Section.

IV. TOPOLOGY INFLUENCE

Let us consider classes of (connected) networks of Hindmarsh-Rose neurons $\mathfrak{N}(N, k)$ defined as the set of networks characterized by a number of nodes N and a node degree k . As seen in Section III, these networks can be synchronized by choosing an appropriate value of η and hence of the coupling strength $g_s = k/\eta$. Furthermore, it is possible to analyze the synchronization of a given network or to synthesize a new network composed of Hindmarsh-Rose neurons with a particular synchronous state by using both the zero-Lyapunov curve of Fig. 2 and the bifurcation diagram of Fig. 3 (see [6] for details and examples.)

Another interesting point of view is that maintaining the *same* coupling strength for all the networks of a class, in general they do not have the same synchronization properties: changing the topology of the network its spectrum changes, and hence, if a network satisfies Equation (7), it may happen that another one does not. In this Section this relation between the topology of the network of Hindmarsh-Rose neurons and the synchronization properties is investigated by analyzing the synchronization behavior of two networks belonging to the same class $\mathfrak{N}(N, k)$, but with different topology.

Let us consider the class of networks $\mathfrak{N}(16, 3)$ and the coupling strength $g_s = 0.4287$. The $\bar{\alpha}$ value is estimated from the zero-Lyapunov exponent curve of Fig. 2: $\bar{\alpha} \simeq 1.06$. It follows that a network synchronizes if and only if

$$g_s \cdot \lambda_2(\mathbf{C}) < \bar{\alpha} \simeq 1.06 \quad (8)$$

The first network is characterized by the topology shown in Fig. 4: the adjacency matrix \mathbf{C} , not reported for lack of space, has been computed numbering clockwise the cells

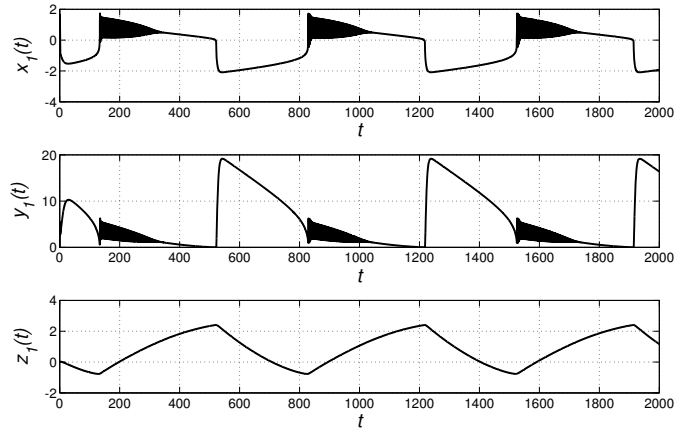


Fig. 5. State evolution of the cell 1 of the network whose topology is reported in Fig. 4.

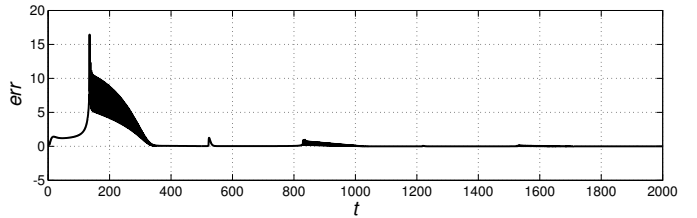


Fig. 6. Quadratic error of the evolution of the network whose topology is reported in Fig. 4.

starting from the black one. The two largest eigenvalues of the associated adjacency matrix are

$$\lambda_1 = 3.0 \quad \lambda_2 = 2.4142 \quad (9)$$

It follows that the considered network synchronizes because

$$g_s \lambda_2(\mathbf{C}) \simeq 1.035 < \bar{\alpha} \simeq 1.06 \quad (10)$$

The evolution of the state variables of the cell 1 and the total quadratic error are reported respectively in Fig. 5 and Fig. 6: the states of the cells synchronize because the quadratic error converge to zero.

The second network is characterized by the topology shown in Fig. 7 in which the cells are numbered following the

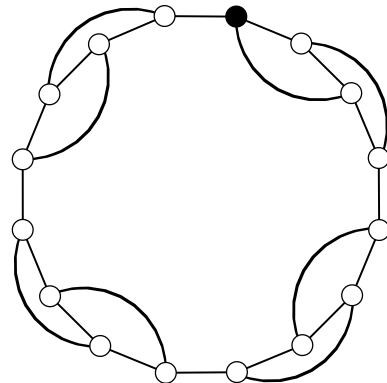


Fig. 7. Topology of a network belonging the class $\mathfrak{N}(16, 3)$. The cells are numbered clockwise starting from the the black one.

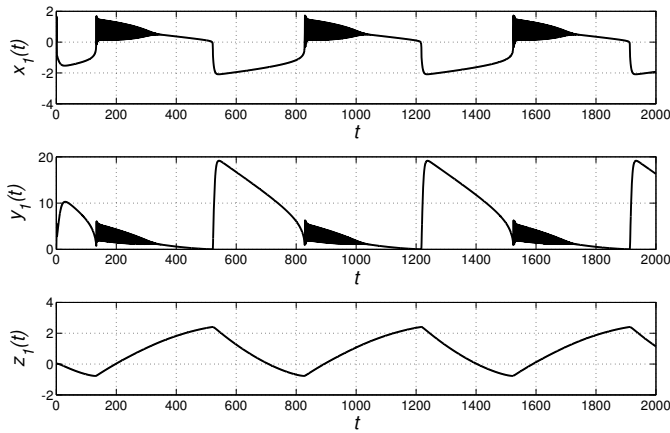


Fig. 8. State evolution of the cell 1 of the network whose topology is reported in Fig. 7.

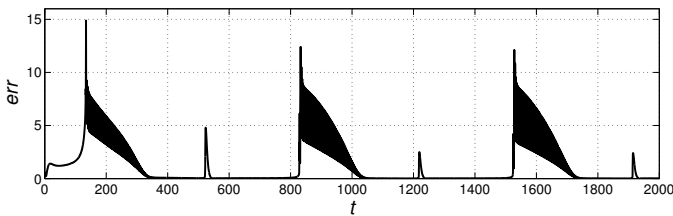


Fig. 9. Quadratic error of the evolution of the network whose topology is reported in Fig. 7.

same rule used for the previous example. The three largest eigenvalues of the associated adjacency matrix are

$$\lambda_1 = 3.0 \quad \lambda_2 = \lambda_3 = 2.7093 \quad (11)$$

It follows that the considered network does not synchronize because

$$g_s \cdot \lambda_2(\mathbf{C}) \simeq 1.161 > \bar{\alpha} \simeq 1.06 \quad (12)$$

The evolution of the states variables of the cell 1 and the total quadratic error are reported respectively in Fig. 8 and Fig. 9: the states of the cells do not synchronize because the quadratic error does not converge to zero. In particular, Figs. 10 and 11 show the evolution of the state of the cells 1 and 9 and the quadratic error, respectively, in a time interval that points out the differences after a long transient.

This example shows that networks belonging to the same class and with the same coupling strength may have different synchronization properties.

V. CONCLUSION

In this work, starting from a method for predicting the behavior of the synchronous state of a network of Hindmarsh-Rose neurons, the dependence of the synchronization properties of a network – keeping constant the number of nodes N , their degree k , and the coupling strength g_s – on its topology has been studied. This dependence has been highlighted by an example.

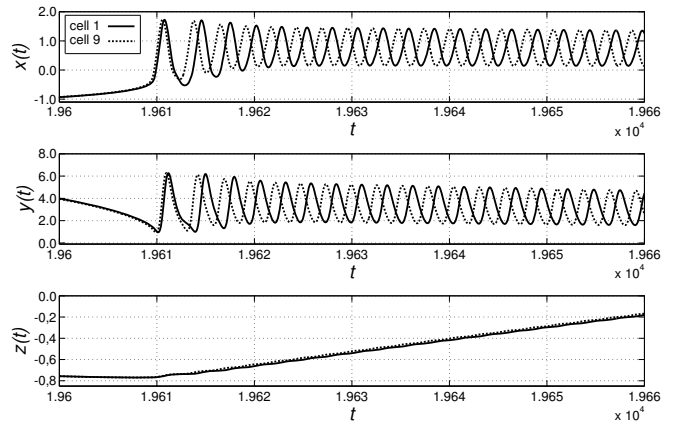


Fig. 10. Enlargement of the state evolution, after the transient, of two cells of the network whose topology is reported in Fig. 7: the solid line is of the cell 1 and the dotted line is of the cell 9.

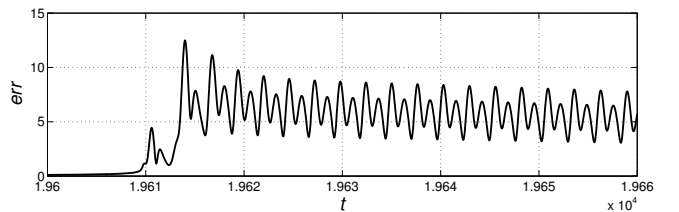


Fig. 11. Enlargement of the quadratic error of the evolution of the network whose topology is reported in Fig. 7 in the same time interval used in Fig. 10.

ACKNOWLEDGEMENT

This work was supported in part by Ministero dell'Università e della Ricerca under PRIN Project no. 2006093814_003.

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