Bifurcation Processes and Chaotic Phenomena in Cellular Neural Networks

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***Abstract* — It is shown that first-order autonomous space-invariant cellular neural networks (CNNs) may exhibit a complex dynamic behavior (i.e. equilibrium point and limit cycle bifurcation, strange and chaotic attractors). The most significant limit cycle bifurcation processes, leading to chaos, are investigated through the computation of the corresponding Floquet's multipliers and Lyapunov exponents. It is worth noting that most practical CNN implementations exploit first order cells and space-invariant templates: so far no example of complex dynamics has been shown in first-order autonomous space-invariant CNNs.**

1 Introduction

Cellular neural networks (CNNs) are analog dynamic processors, composed of identical nonlinear dynamical systems (called cells), that are locally interconnected. They have found important applications for the solution of several complex computational problems [1, 2, 3]. In most applications the connections are specified through space-invariant templates.

The mathematical model of a CNN consists in a large set of coupled nonlinear differential equations, that have been mainly studied through extensive computer simulations.

For what concerns the dynamic behavior, CNNs can be divided in two classes: stable CNNs, with the property that each trajectory converges towards an equilibrium point; unstable CNNs, that exhibit at least one attractor, that is not a stable equilibirum point. The stability results are summarized in [4], whereas some examples of CNNs presenting periodic limit cycles are shown in [5, 6].

Complex dynamics in CNNs (i.e. networks presenting non-periodic, possibly strange, attractors) have been so far observed only in four cases: a) non-autonomous networks composed by two cells [7]; b) autonomous CNNs described by space-variant templates and composed by three cells [8]; c) delayed CNNs [9]; d) state-controlled CNNs [10]. So far, no example of complex dynamics has been shown in first-order autonomous CNNs, described by space invariant templates.

The importance of investigating the complex dynamic behavior of first-order autonomous space-invariant CNNs relies on the fact that such a model is exploited by most VLSI implementations [3]. The identification of chaotic dynamics in these networks might open the possibility of developing, on the existing CNN chips, innovative chaos-based applications.

In this paper we show that, for a class of two-dimensional opposite-sign templates, complex dynamic occurs. Starting from a stable CNN, we investigate the equilibrium point bifurcation, leading to periodic attractors. Then we study the limit cycle bifurcation route to chaos, through the computation of the corresponding Floquet's multipliers and Lyapunov exponents.

2 Space-invariant CNNs

We consider CNNs composed by $N \times M$ cells arranged on a regular grid and denote the position of a cell with two indexes (k, l).

The network dynamics is described by the following normalized state equations

$$\begin{aligned} \dot{x}_{kl} &= -x_{kl} + \sum_{|n|\le r,|m|\le r} A_{nm}\, y_{k+n,l+m} \\ &\quad + \sum_{|n|\le r,|m|\le r} B_{nm}\, u_{k+n,l+m} + I \qquad (1) \end{aligned}$$

where x_{kl} and u_{kl} represent the state-voltage and the input voltage of cell (k, l). We assume that the output voltage y_{kl} is defined through the function $f_\varepsilon(x)$ shown in Fig. 1, that is a smooth version of the original Chua-Yang piecewise linear output function. Function $f_\varepsilon(x)$ depends on the parameter ε; it presents the advantage of being continuous with its first-order derivative and therefore to allow an accurate analysis of limit cycle bifurcations.

Finally, r denotes the neighborhood of interaction of each cell; A_{nm} and B_{nm} are the elements of the linear templates $\boldsymbol{A}$ and $\boldsymbol{B}$, that are assumed to be space-invariant, and I is the bias term.

We remark that the qualitative dynamics of the above model is very similar to that of the models adopted for VLSI implementation (see [3]).

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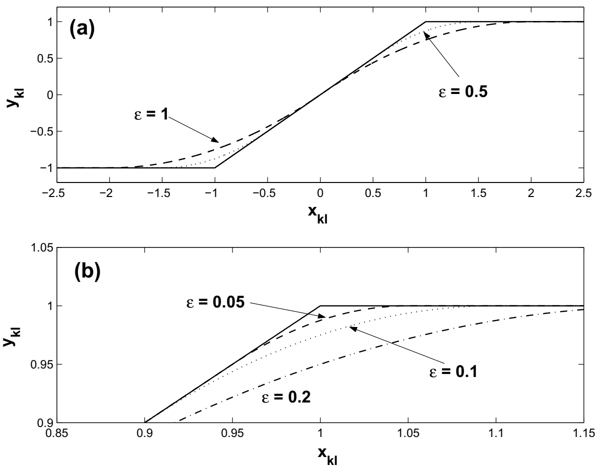

Figure 1: *Comparison between the smooth function $f_\varepsilon(\cdot)$ and the Chua-Yang piecewise linear function; (a) $\varepsilon = 1,\ 0.5$; (b) zoom in the neighborhood of $x_{kl} = 1$, for $\varepsilon = 0.2,\ 0.1,\ 0.05$.*

3 Complex dynamics

We consider an autonomous CNN composed by 3×3 cells, with zero boundary conditions and zero input voltages, and described by the following opposite-sign $\boldsymbol{A}$ template

$$\boldsymbol{A} = \begin{bmatrix} a & a & a \\ -a & b & a \\ -a & a & -a \end{bmatrix} \quad \text{with } b > 1;\ \ a > 0 \qquad (2)$$

We assume that the parameter ε, describing the output function $f_\varepsilon(\cdot)$, is set to 0.1.

The above class of templates (2) exhibits the following property: by increasing a all the stable equilibrium points disappear and therefore the network should present at least a non-stationary attractor (i.e. either a periodic or a non-periodic attractor).

We assume $b = 1.6$ and investigate the dynamics of the network and the related bifurcation processes, that can be observed by varying a.

For $a = 0$ the cells are not coupled and each of them presents two stable equilibrium points, with output voltage $y_{kl} = \pm 1$ respectively. The whole network exhibits 2^9 stable equilibrium points. We have verified that, by increasing a, all the stable equilibrium points undergo a saddle-node bifurcation; for $a = 0.575$ all the stable equilibria disappear and the CNN presents a stable limit cycle (that will be denoted as c_1). Such a cycle probably originates through an heteroclinic bifurcation, since its period increases and tends to a vertical asymptot, as a approaches the value 0.575 (see Fig. 2).

We have computed the Floquet's multipliers

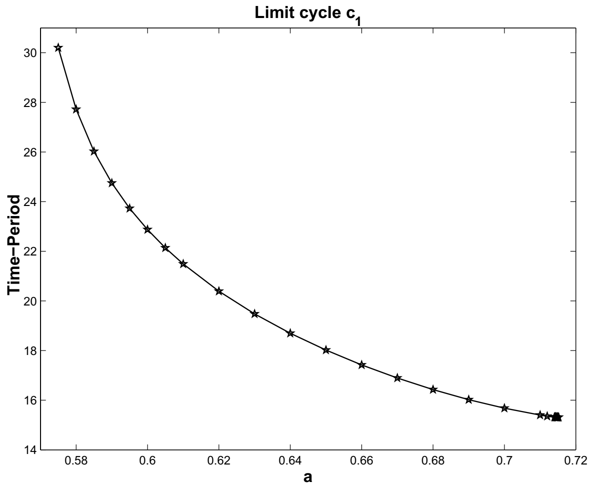

Figure 2: *Limit cycle c_1: time period versus a.*

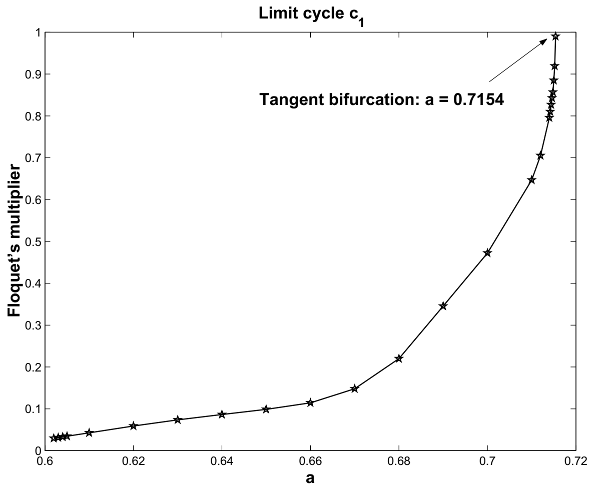

Figure 3: *Limit cycle c_1: most significant FM, versus a.*

(FMs) of cycle c_1, by exploiting the algorithm reported in [12]. Apart from the structural unitary FM, only one FM is significantly different from zero and therefore determines the limit cycle bifurcation: such a FM is reported in Fig. 3. It is seen that for $a = 0.7154$ this FM reaches the value 1, i.e the limit cycle c_1 disappear through tangent bifurcation. The projection of the steady-state trajectory onto the plane $(x_{11},\ x_{12})$ is shown in Fig. 4, for some values of the parameter a within the range of existence of c_1, i.e. $a \in [0.6,\ 0.7154]$.

The simulation shows that for $a > 0.65$ another stable limit cycle coexists with c_1: such a cycle is denoted with c_2. The most significant FM associated to c_2 is reported in Fig. 5, as a function of a. The analysis of the FMs shows that c_2 originates through a tangent bifurcation (as a approaches 0.65

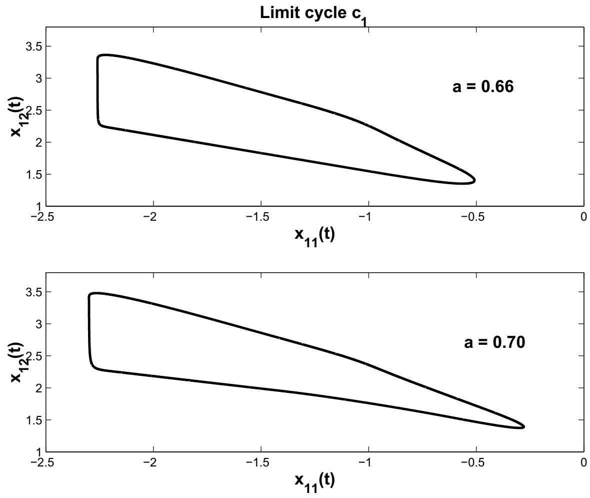

Figure 4: *Limit cycle* c_1*: projection of the trajectory onto the plane* (x_{11}, x_{12}).

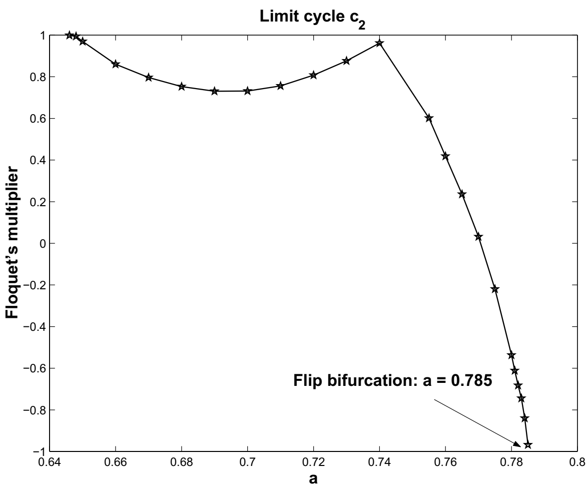

Figure 5: *Limit cycle* c_2*: most significant FM, versus* a.

from the right, one FM approaches 1); by increasing a, one FM equals -1 and this reveals that c_2 undergoes a typical period doubling (flip) bifurcation, for $a = 0.785$. As a result of this bifurcation, c_2 becomes unstable and a new cycle (denoted with c_4) of period approximately twice arises (see Fig. 6). The main characteristics of these two cycles are reported in Fig. 7.

By further increasing a, a sequence of period doubling bifurcation leading to a chaotic attractor is observed (see Fig.8, for $a = 0.8$). The Lyapunov exponents associated to the cycles c_2 (for $a = 0.74$) and to the chaotic attractor, obtained for $a = 0.8$, have been computed, by exploiting the software tool described in [13]; they are reported in Table 1 and confirm that the chaotic attractor presents a positive Lyapunov exponents.

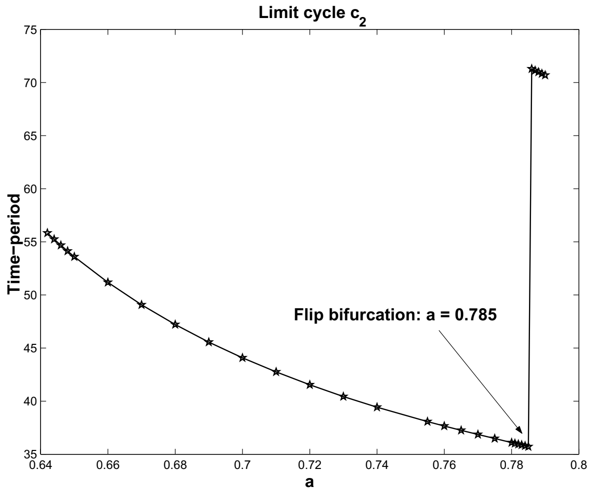

Figure 6: *Limit cycle* c_2*: time period versus* a.

It is worth noting that, for the above CNN, the system of equation (1) is invariant under the following coordinate transformations:

$$
\begin{array}{ll}
\mathcal{T} \;\rightarrow & \left\{ \begin{array}{lll} z_{11} = -x_{33} & z_{12} = x_{32} & z_{13} = -x_{31} \\ z_{21} = -x_{23} & z_{22} = x_{22} & z_{23} = -x_{21} \\ z_{31} = -x_{13} & z_{32} = x_{12} & z_{33} = -x_{11} \end{array} \right. \\
\mathcal{O} \;\rightarrow & \qquad\qquad z_{ij} = -x_{ij}
\end{array}
\tag{3}
$$

It turns out that if the CNN presents an invariant set l (i.e. limit cycles c_1, c_2, c_4 and the chaotic attractor of Fig. 8), then it should also exhibit: a) the limit set obtained by applying to l the coordinate transformation $\mathcal{T}(l)$; the two limit sets symmetric to l and $\mathcal{T}(l)$ with respect to the origin, i.e. $\mathcal{O}(l)$ and $\mathcal{O}[\mathcal{T}(l)]$.

Lyapunov exponents	
Limit cycle c_2	Chaotic attractor
0.000833	0.036529
-0.016316	0.000034
-0.587443	-0.493471
-0.665539	-0.725800
-0.833612	-0.802559
-0.876804	-0.838617
-0.890424	- 0.869474
-0.931914	-0.919737
-0.989583	-0.995783

Table 1: Lyapunov exponents for the limit cycle c_2 ($a = 0.74$) and the chaotic attractor, observed for $a = 0.8$.

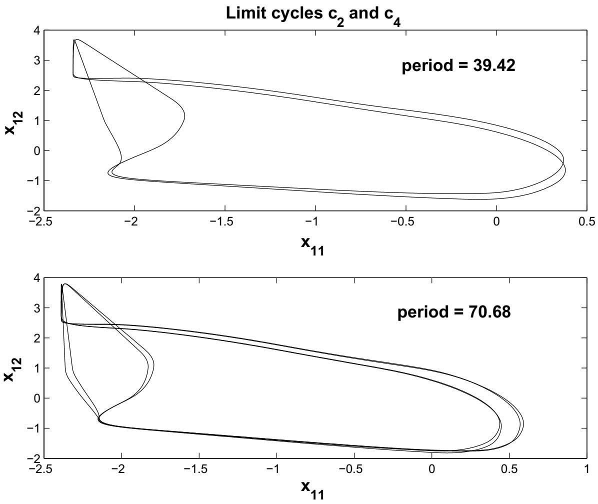

Figure 7: *Projection of the trajectory onto the plane* (x_{11}, x_{12}) *for the limit cycles* c_2 *(upper figure,* $a = 0.74$*) and the limit cycle* c_4 *(lower figure,* $a = 0.79$*).*

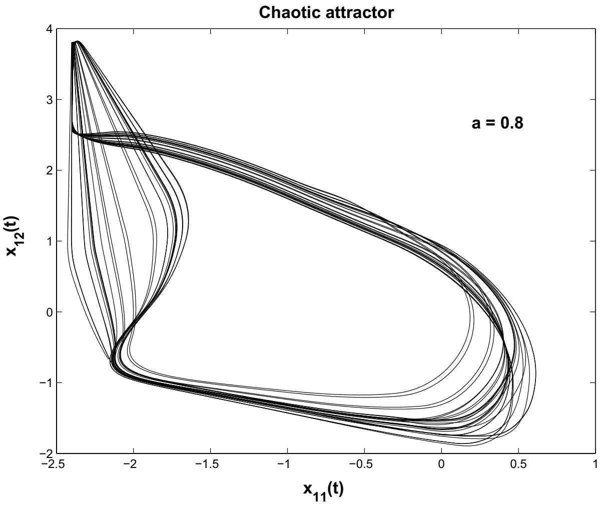

Figure 8: *Chaotic attractor, originated through the bifurcation of cycle* c_2*: projection of the trajectory onto the plane* (x_{11}, x_{12}) *for* $a = 0.8$ *.*

4 Conclusions

We have investigated the occurrence of complex dynamic behaviors (i.e. bifurcation processes, strange and chaotic attractors) in first-order autonomous space-invariant CNNs. There are some reasons for carrying on this study: a) most CNN implementation exploits space-invariant templates; b) so far no example of complex dynamics has been shown in first-order autonomous space-invariant CNNs. Hence, we are confident that the identification of chaotic dynamics in such networks might open the possibility of developing, on the existing CNN chips, innovative chaos-based applications.

Starting from a first-order autonomous CNN, described by a two-dimensional space-invariant template, we have investigated the equilibrium point bifurcation, leading to periodic attractors. Then we have studied in detail the limit cycle bifurcation route to chaos, through the computation of the corresponding Floquet's multipliers and Lyapunov exponents.

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