Available online at www.sciencedirect.com
ScienceDirect
Chaos, Solitons and Fractals 35 (2008) 562–577
CHAOS
SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Synchronization in random networks with given expected degree sequences

Paolo Checco [a,*], Mario Biey [a], Ljupco Kocarev [b]

[a] *Dipartimento di Elettronica, Politecnico di Torino, Torino, Italy*
[b] *Institute for Nonlinear Science, University of California, San Diego, La Jolla, CA, USA*

Accepted 22 May 2006

Abstract

Synchronization in random networks with given expected degree sequences is studied. We also investigate in details the synchronization in networks whose topology is described by classical random graphs, power-law random graphs and hybrid graphs when $N \to \infty$. In particular, we show that random graphs almost surely synchronize. We also show that adding small number of global edges to a local graph makes the corresponding hybrid graph to synchronize.

1. Introduction

The study of complex systems pervades all of science, from cell biology to ecology, from computer science to meteorology. A paradigm of a complex system is a network [1], where complexity may come from different sources: topological structure, network evolution, connection and node diversity, and/or dynamical evolution. Examples of networks include food webs [2,3], electrical power grids, cellular and metabolic networks, the World-Wide Web [4], the Internet backbone [5], neural networks, and co-authorship and citation networks of scientists. These networks consist of nodes which are interconnected by a mesh of links. The macroscopic behavior of a network is determined by both the dynamical rules governing the nodes and the flow occurring along the links.

Real networks of interacting dynamical systems – be they neurons, power stations or lasers – are complex. Many real-world networks are small-world [6] and/or scale-free networks [7]. The presence of a power-law connectivity distribution, for example, makes the Internet a scale-free network. The research on complex networks has been focused so far on the their topological structure [8]. However, most networks offer support for various dynamical processes. In this paper we propose to study one aspect of dynamical processes in non-trivial complex network topologies, namely their synchronization behavior.

The general question of network synchronizability, for many aspects, is still an open and outstanding research problem [9,10]. There are, in general, two classes of results which give criteria under which a network of oscillators synchronizes. The first class of results uses Lyapunov's direct method by constructing a Lyapunov function which decreases along trajectories and gives analytical criteria for local or global synchronization. For example, in [11], the authors gave

* Corresponding author.
E-mail addresses: paolo.checco@polito.it (P. Checco), mario.biey@polito.it (M. Biey), lkocarev@ucsd.edu (L. Kocarev).

0960-0779/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.chaos.2006.05.063

sufficient conditions for an array of linearly coupled systems to synchronize. A typical result states that the array will synchronize if the non-zero eigenvalues of the coupling matrix have real parts that are negative enough. The work in [11] has been extended and generalized in [12–16].

The second class of results uses linearized equations around the synchronization manifold and computes numerically the Lyapunov exponents of the variational equations. In this context, an important contribution has been given by Pecora and Carroll in [17], where, for a network of coupled chaotic oscillators, they derived the so-called *Master Stability Equation* (MSE), and introduced the corresponding *Master Stability Function* (MSF). Consequently, the stability analysis of the synchronous manifold [17] for the network under consideration can be decomposed in two sub-problems. The first sub-problem consists of deriving the MSF for the network nodes, i.e. to study in which region, of the complex plane the MSE admits a negative largest Lyapunov exponent (LE). The second sub-problem is to verify whether the eigenvalues of the so-called *connectivity matrix* [18] of the network, apart from the zero-eigenvalue, lie in the synchronization region(s) (see also [17–19]). This approach is particularly relevant because the MSE depends only on the node local dynamics and on the *coupling matrix* [18]. It turns out that the mathematical problem has the same dimension as the single network node. For example, when considering a network of coupled Rössler systems [20], the master stability equation has dimension three.

Recently, the synchronization phenomenon in scale-free dynamical networks has been investigated in [21–25]. In [22], the authors found that networks with a homogeneous distribution of connectivity are more synchronizable than heterogeneous ones, even though the average network distance is larger. They presented numerical computations and analytical estimates on synchronizability of the network in terms of its heterogeneity parameters. Robustness and/or fragileness of the networks' synchrony is discussed in [21,23,24]. Network synchronization and de-synchronization processes in a scale-free network are illustrated by a prototype composing of Henon maps. A new general method to determine global stability of total synchronization in networks with different topologies is proposed in [26,27]. This method combines the Lyapunov function approach with graph theoretical reasoning. In particular, the method is applied to the study of synchronization in rings of $2K$-nearest neighbor coupled oscillators. This method is extended to the blinking model of small-world networks where, in addition to the fixed $2K$-nearest neighbor interactions, all the remaining links are rapidly switched on and off independently of each other. In [25], the authors studied synchronization in weighted complex networks and showed that the synchronizability of random networks with a large minimum degree is determined by two leading parameters: the mean degree and the heterogeneity of the distribution of node intensity, where the intensity of a node, defined as the total strength of input connections, is a natural combination of topology and weights.

In this work, following [17], we first review the properties of the master stability function. Namely, in Section 2 it is shown that for typical systems only three main scenarios may arise as a function of coupling strength: the synchronization region may have the following forms: an interval $(\alpha_m, +\infty)$, union of intervals $(\alpha_m^{(j)}, \alpha_M^{(j)})$, or an empty set. Then, we study synchronization in complex networks topologies. Section 3 is devoted to the analysis of synchronization properties of networks whose topology is described by classical and power-law random graph models. We prove that random graph networks synchronize. In this paper we consider the model $M(N,\beta,d,m)$, where N is the number of vertices, $\beta > 2$ is the power of the power law, d is the expected average degree, and m is the expected maximum degree, such that $m^2 = o(Nd)$. We prove the following theorem: Let $M(N,\beta,d,m)$ be a random power-law graph on N vertices, for which d grows with N. Assume further that d/m approaches 0 when $N \to \infty$. Then the class-A network $M(N,\beta,d,m)$ asymptotically almost surely synchronizes for arbitrary small coupling σ and class-B network $M(N,\beta,d,m)$ asymptotically almost surely does not synchronize. In Section 4, we study synchronization properties of hybrid networks. We prove that although local graph networks do not synchronize for large N, adding only a small number of global edges makes these hybrid networks to synchronize. We close our paper with conclusions.

2. Preliminaries: master stability function

In this section, following [17], we first review the properties of the master stability function.

2.1. Introduction

Let us consider a network comprising N identical nodes, each being a (chaotic) oscillator. Let $\boldsymbol{x}_i$ be the m-dimensional vector of dynamical variables for the ith node. Let the dynamics of each node be described by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i) + \sum_{k=1}^{N} \boldsymbol{D}_{ik}\boldsymbol{x}_k, \quad i = 1,\ldots,N, \tag{1}$$

where $\boldsymbol{f}:\mathbb{R}^m\to\mathbb{R}^m$ describes the oscillator equations, which we assume to admit a chaotic attractor [20], while $\boldsymbol{D}_{ik}$ are $m\times m$ real matrixes. Assume that each matrix, $\boldsymbol{D}_{ik}$, has the form: $\boldsymbol{D}_{ik}=g_{ik}\boldsymbol{H}$, where g_{ik} is a real number and $\boldsymbol{H}$ is a $m\times m$ diagonal matrix, same for all nodes, called *coupling matrix*. The coupling matrix $\boldsymbol{H}=(h_{ij})$ contains the information about which variables are utilized in the coupling and is defined as $h_{ii}=1$, if the ith component is coupled, and $h_{ii}=0$, otherwise. Let us denote with $\boldsymbol{G}=(g_{ij})$ an $N\times N$ matrix called the *connectivity matrix*, because it specifies which nodes are connected to which and the coupling strength. Recall that the direct product of two matrixes $\boldsymbol{A}=(a_{ij})$ and $\boldsymbol{B}$ is given, in block form, by

$$\boldsymbol{A}\otimes\boldsymbol{B}=\begin{pmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \dots & a_{1N}\boldsymbol{B}\\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \dots & a_{2N}\boldsymbol{B}\\ \vdots & \vdots & \vdots & \vdots\\ a_{N1}\boldsymbol{B} & a_{N2}\boldsymbol{B} & \dots & a_{NN}\boldsymbol{B}\end{pmatrix}.$$

Then, we can rewrite Eq. (1) in a more compact form using the direct product of matrixes

$$\dot{\boldsymbol{x}}=\mathbf{F}(\boldsymbol{x})+(\boldsymbol{G}\otimes\boldsymbol{H})\boldsymbol{x}, \tag{2}$$

where $\mathbf{F}(\boldsymbol{x}):\mathbb{R}^{mN}\to\mathbb{R}^{mN}$, $\boldsymbol{x}=(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)^{\mathrm{T}}$ and is defined as $\mathbf{F}(\boldsymbol{x})=(\boldsymbol{f}(\boldsymbol{x}_1),\dots,\boldsymbol{f}(\boldsymbol{x}_N))^{\mathrm{T}}$. The $(N-1)$ constraints: $\boldsymbol{x}_1=\boldsymbol{x}_2=\cdots=\boldsymbol{x}_N$, define the so-called *synchronization manifold* [17]. The invariance of this manifold requires that: $\sum_j g_{ij}=0$, $\forall i$. To determine the stability of the synchronization manifold, one should evaluate the Lyapunov exponents along the directions transverse to the manifold itself. In this respect, the variational equation of the system (2) is

$$\dot{\boldsymbol{\xi}}=[\boldsymbol{J}_F+\boldsymbol{J}_{G\otimes H}]\xi, \tag{3}$$

where $\boldsymbol{\xi}=(\xi_1,\xi_2,\dots,\xi_N)^{\mathrm{T}}$ and $\boldsymbol{J}_F$ and $\boldsymbol{J}_{G\otimes H}$, are the Jacobian matrixes of $\mathbf{F}$ and $\boldsymbol{G}\otimes\boldsymbol{H}$, respectively. By noting that the matrixes $\boldsymbol{G}$ and $\boldsymbol{H}$ do not depend on $\boldsymbol{x}$ and on the synchronization manifold, then $\boldsymbol{J}_F=\boldsymbol{I}_N\otimes\boldsymbol{J}_f$, where $\boldsymbol{I}_N$ is the $N\times N$ identity matrix. It follows that Eq. (3) can be rewritten as:

$$\dot{\boldsymbol{\xi}}=[\boldsymbol{I}_N\otimes\boldsymbol{J}_f+\boldsymbol{G}\otimes\boldsymbol{H}]\boldsymbol{\xi}. \tag{4}$$

Since $\boldsymbol{G}$ is a constant $N\times N$ matrix, the matrix that diagonalizes $\boldsymbol{G}$ is also constant and can be applied directly to (4). Thus, we have

$$\dot{\zeta}_k=[\boldsymbol{J}_f+\gamma_k\boldsymbol{H}]\zeta_k \tag{5}$$

with $k=1,\dots,N$ and γ_k are the eigenvalues of the connectivity matrix $\boldsymbol{G}$. Note that, for $k=1$, Eq. (5) is the variational equation of the synchronization manifold, i.e. the eigenvalue $\gamma_1=0$. On the other hand, the other values of $k>1$ correspond to all transverse eigenvectors. The two matrixes in (5), $\boldsymbol{J}_f$ and $\boldsymbol{H}$, are constants with respect to k, and only the eigenvalues γ_k vary. Thus, one can reformulate the above equation as follows:

$$\dot{\zeta}=[\boldsymbol{J}_f+(\alpha+\imath\beta)\boldsymbol{H}]\zeta, \tag{6}$$

that is the *master stability equation*. This equation depends on the two parameters α and β, and the corresponding largest Lyapunov exponent, which is also a function of α and β, represents the *master stability function*.

2.2. Properties of the master stability function

We now study the properties of MSF. Let us write the master stability equation (6) as

$$\dot{\zeta}=\boldsymbol{J}\zeta, \tag{7}$$

where $\boldsymbol{J}=[\boldsymbol{J}_f+(\alpha+\imath\beta)\boldsymbol{H}]$. In the following, we denote with $\Phi_{\boldsymbol{J}}(t,t_0)$ the *transition matrix* of the system (7) (see [28]). The MSE (7) is defined by the matrix $\boldsymbol{J}$ which, in turn, comprises two terms. The first term is given by the Jacobian matrix $\boldsymbol{J}_f$ of $\boldsymbol{f}$, that is a real $m\times m$ matrix, for ordinary systems. The second term is the real matrix, $\boldsymbol{H}$, multiplied by $(\alpha+\imath\beta)$, which is a complex number. Therefore, the matrix $\boldsymbol{J}$ and the transition matrix $\Phi_{\boldsymbol{J}}(t,t_0)$ of MSE (7) are complex. The m transition matrix eigenvalues are complex (but not conjugate, in general, since the matrix is not real).

We recall the definition of Lyapunov exponents

$$\lambda_i=\lim_{t\to\infty}\frac{1}{t}\ln|m_i(t)|, \tag{8}$$

where $m_i(t)$ denotes the eigenvalues of the transition matrix (solution of the dynamical system). We observe that, according to the definition (8) of Lyapunov exponents, only the absolute value of the eigenvalues, m_i, comes into play.

Thus, the Lyapunov exponents are the same for $(\alpha + \imath\beta)$ and its conjugate. So, it follows that the master stability function is symmetric with respect to the real α-axis.

*2.2.1. Coupling matrix **H** equal to the identity matrix*

The study of the properties of the MSF can be subdivided into two cases, depending on whether the connectivity matrix $\boldsymbol{H}$ coincides with the identity matrix or not. In the former case, the Jacobian matrix $\boldsymbol{J}$, used for evaluating the Lyapunov exponents, is given by

$$\boldsymbol{J} = \boldsymbol{J}_f + (\alpha + \imath\beta)\boldsymbol{I}. \tag{9}$$

Proposition 1. *If the coupling matrix $\boldsymbol{H}$ is equal to the identity matrix $\boldsymbol{I}$, then the Lyapunov exponents $\lambda_i(\alpha, \beta)$ of the master stability equation (7) are*

$$\lambda_i(\alpha, \beta) = \lambda_i(0, 0) + \alpha. \tag{10}$$

Proof. In order to prove Eq. (10), we consider the following linear time-variant differential equation (of order m):

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi_A(t, t_0) = A(t)\Phi_A(t, t_0), \\ \Phi_A(t_0, t_0) = I, \end{cases} \tag{11}$$

where $\Phi_A(t, t_0)$ is the transition matrix. Its solution is given by the Peano–Baker series [28]

$$\Phi_A(t, t_0) = I + \int_{t_0}^{t} \mathrm{d}t_1 A(t_1) + \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 A(t_1)A(t_2) + \cdots. \tag{12}$$

Then, we can rewrite the solution as

$$\Phi_A(t, t_0) = I + \int_{t_0}^{t} \mathrm{d}t_1 T\{A(t_1)\} + \frac{1}{2!}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 T\{A(t_1)A(t_2)\} + \cdots \triangleq T\left\{\mathrm{e}^{\int_{t_0}^{t} \mathrm{d}t_1 A(t_1)}\right\}, \tag{13}$$

where $T\{\cdot\}$ is the time order-product

$$\begin{cases} T\{A(t_1)A(t_2)\cdots A(t_m)\} = A(t_{i_1})A(t_{i_2})\cdots A(t_{i_m}), \\ t_{i_1} \geqslant t_{i_2} \geqslant \cdots \geqslant t_{i_m}, \\ \{t_{i_1}, t_{i_2}, \ldots t_{i_m}\} = \{t_1, t_2, \ldots t_m\}. \end{cases} \tag{14}$$

From Eq. (13), it is straightforward to verify that the following equality holds true:

$$T\left\{\mathrm{e}^{\int_{t_0}^{t} \mathrm{d}t_1 (A(t_1)+cI)}\right\} = T\left\{\mathrm{e}^{\int_{t_0}^{t} \mathrm{d}t_1 A(t_1)}\right\} \cdot T\left\{\mathrm{e}^{c\int_{t_0}^{t} \mathrm{d}t_1 I}\right\} = T\left\{\mathrm{e}^{\int_{t_0}^{t} \mathrm{d}t_1 A(t_1)}\right\} \cdot \mathrm{e}^{c(t-t_0)}. \qquad \square \tag{15}$$

Note that the largest Lyapunov exponent depends only on the (largest of the) LEs of the original dynamical system $\boldsymbol{f}$, and on α. The MSF is then a plane with slope equal to α, and it does not depend on the value of β. In particular, the MSF is equal to zero for $\alpha = -\lambda_{\max}(0, 0)$ and negative in the left half-plane, with respect to the latter straight line.

*2.2.2. Coupling matrix **H** not equal to the identity matrix*

In this case, we use an asymptotic method with respect the parameters α and β to estimate the largest Lyapunov exponent of the master stability equation. It is easy to see that, if the parameter α is positive and the absolute value $|\alpha + \imath\beta|$ tends to infinity, the largest Lyapunov exponent is approximatively equal to α. If α is negative, with the same condition for the above absolute value, the largest Lyapunov exponent is constant

$$\lambda_{\max}(\alpha, \beta) \to \begin{cases} \alpha & \text{if } \alpha > 0, \\ r & \text{if } \alpha < 0, \end{cases} \quad |\alpha + \imath\beta| \to \infty, \tag{16}$$

where r is the conditional LE [18], constant with respect α and β.

We remark that (16) is true for $|\alpha + \imath\beta|$ tending to infinity. In practice, this condition is met provided that $|\alpha + \imath\beta|$ is sufficiently larger than all entries in the transition matrix of the variational equation, for a single node (i.e. the master stability equation with parameters $\alpha = \beta = 0$). This fact has been confirmed by computing the master stability function for several different dynamical systems. This also means that, if $|\alpha + \imath\beta|$ is sufficiently large (with respect to all entries of

$\Phi_J(t,t_0)$ in the case α and β equal to zero), the MSF is constant if α is negative and approaches a plane with slope equal to α, otherwise. Moreover, the MSF depends on β only when the condition $|\alpha+\imath\beta|\to\infty$ is not fulfilled.

2.3. Synchronization regions

We now consider the following form of Eq. (2):

$$\dot{\boldsymbol{x}} = \mathbf{F}(\boldsymbol{x}) + \sigma(\boldsymbol{L}\otimes\boldsymbol{H})\boldsymbol{x}, \tag{17}$$

where σ is the overall strength of coupling, and the $N\times N$ matrix $\boldsymbol{L}$ is the Laplacian matrix representing the connection topology of the network: $l_{ij}=l_{ji}=-1$ if nodes i and j are connected, $l_{ii}=k_i$ if node i is connected to k_i other nodes, and $l_{ij}=l_{ji}=0$ otherwise.

The matrix $\boldsymbol{L}$, which will be our main concern, is positive semi-definite and symmetric. Recall $\gamma_1=0\leqslant\gamma_2\leqslant\cdots\leqslant\gamma_N$, repeated according to their multiplicities, are eigenvalues of the matrix L. In particular, γ_N, is the maximal eigenvalue of $\boldsymbol{L}$.

Since $\boldsymbol{L}$ is symmetric, the master stability function, in this case, has the form

$$\dot{\zeta} = [\boldsymbol{J}_f + \alpha\boldsymbol{H}]\zeta, \tag{18}$$

where $\alpha\in\mathbb{R}$. Therefore, in this case the corresponding largest Lyapunov exponent or MSF, $\Lambda(\alpha)$, depends only on one parameter, α. The master stability function determines the linear stability of the synchronized state; in particular, the synchronized state is stable if all the eigenvalues of the matrix $\boldsymbol{L}$ (apart the zero one) are in the region $\Lambda(\alpha)<0$. We denote by $S\subseteq\mathbb{R}$ the region where the MSF is negative and call it *synchronization region*. Discussions in the previous sections show in fact that for the system (17), the synchronization region S may have one of the following forms:

- $S_1=\emptyset$.
- $S_2=(\alpha_m,+\infty)$.
- $S_3=\bigcup_j(\alpha_m^{(j)},\alpha_M^{(j)})$.

Examples of the these scenarios are given in [29,30]. In the majority of cases α_m, $\alpha_m^{(j)}$, and $\alpha_M^{(j)}$ turn out be positive and, furthermore, in the case S_3 there is only one parameter interval $\left(\alpha_m^{(j)},\alpha_M^{(j)}\right)$ on which $\Lambda(\alpha)<0$. For this reason, we will limit ourself to consider only such cases, focusing, in the remaining of this paper, on the scenarios $S_2=(\alpha_m,+\infty)$ and $S_3=(\alpha_m,\alpha_M)$. It is easy to see that for S_2 the condition of stable synchronous state is $\sigma\gamma_2>\alpha_m$. For S_3, one can easily show that there is a value of the coupling strength σ for which the synchronization state is linearly stable, if and only if $\gamma_N/\gamma_2<\alpha_M/\alpha_m$. Therefore, for a large class of (chaotic) oscillators there exist two classes of networks:

1. Class-A networks: networks whose synchronization region is of type S_2, for which the condition of stable synchronous state is $\sigma\gamma_2>a$;
2. Class-B networks: networks whose synchronization region is of type S_3, for which this condition reads $\gamma_N/\gamma_2<b$;

Table 1
Examples of class-A and class-B networks

$\boldsymbol{H}$			Network Class Type	Synchronization region	
h_{11}	h_{22}	h_{33}		Case 1	Case 2
1	0	0	Class-A	$\alpha>5.14$	$\alpha>5.83$
0	1	0	Class-A	$\alpha>0.00$	$\alpha>0.86$
0	0	1	Class-B	$1.52<\alpha<2.48$	$1.81<\alpha<2.50$
1	1	0	Class-A	$\alpha>0.00$	$\alpha>0.90$
1	0	1	Class-A	$\alpha>0.00$	$\alpha>0.15$
0	1	1	Class-A	$\alpha>0.00$	$\alpha>1.13$
1	1	1	Class-A	$\alpha>0.00$	$\alpha>0.13$

The Chua's circuit is used as an oscillator in two different cases: Case 1, for which the circuit has an attracting limit cycle, and Case 2, for which the circuit shows a chaotic attractor. For each case, all 7 different types of coupling matrix have been investigated (note that $h_{ij}=0$ for $i\neq j$).

where $a = \alpha_m$ and $b = \alpha_M/\alpha_m$ are constants that depend on $\boldsymbol{f}$, the synchronous state $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \cdots = \boldsymbol{x}_N$ and the matrix $\boldsymbol{H}$, but not on the Laplacian matrix $\boldsymbol{L}$. For typical oscillators $b > 1$.

An example, showing class-A networks and class-B networks, is given in Table 1. In this table, the Chua's circuit is used as an oscillator (see [31]) and all different forms of the connectivity matrix $\boldsymbol{H}$ are considered. The values of parameters for the Case 1 are $\alpha = 8.0$, $\beta = 100/7$, $m_0 = -8/7$, and $m_1 = -5/7$, while the parameters for the Case 2 are same as for the Case 1, except for $\alpha = 8.5$. Note that for the Case 2, a takes values in the range $[0.13, 5.83]$, while $b = 1.38$. Other examples include networks of Lorenz oscillators (class-A network), for which $a = 0.908$, and networks of Rossler oscillators (class-B network), for which $b = 37.85$.

3. Synchronization in random graphs

3.1. Preliminaries

A *graph* is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of unordered pairs of V. The set V is the set of *vertices* and E is the set of *edges*. If G is a graph then $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set. The edge $\{v_i, v_j\}$ is said to *join* the vertices v_i and v_j and is denoted by v_iv_j. Thus v_iv_j and v_jv_i means exactly the same edge; the vertices v_i and v_j are the *endvertices* of this edge. If $v_iv_j \in E(G)$ then v_i and v_j are *adjacent* or *neighboring* vertices of G and the vertices v_i and v_j are *incident* with edge v_iv_j.

The *order* of G is the number of vertices; it is denoted by $|G|$, where $|\cdot|$ denotes the number of elements (cardinality) of a set. The *size* is the number of edges; it is denoted by $e(G)$. We write G_N for an *arbitrary graph of order N*. Similarly $G(N, m)$ denotes an *arbitrary graph of order N and size m*.

The set of vertices adjacent to a vertex $v_i \in G$ is denoted by $\Gamma(v_i)$. The *degree* of v_i is $d(v_i) = |\Gamma(v_i)|$. The *minimum degree* of the vertices of a graph G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. If $\delta(G) = \Delta(G) = k$, that is every vertex of G has degree k then G is said to be *k-regular* graph. If $V(G) = \{v_1, v_2, \ldots, v_N\}$, then $\delta(G) = d(v_1) \leqslant d(v_2) \leqslant \cdots \leqslant d(v_N) = \Delta(G)$ is a *degree sequence* of G. The average degree or simply degree of a graph is $d(G) = \sum_i d(v_i)/N = 2e(G)/|G|$. The degree distribution $p_d(k)$ denotes the fraction of vertices that have degree equal to k.

The size of a graph of order N is at least 0 and most $N(N-1)/2$. Clearly for every m, $0 \leqslant m \leqslant N(N-1)/2$, there is a graph $G(N, m)$. A graph of order N and size $N(N-1)/2$ is called a *complete n-graph* and is denoted by K_N. A *path* is a graph P of the form

$$(V(P), E(P)) : V(P) = \{v_0, v_1, \ldots, v_l\}, \quad E(P) = \{v_0v_1, v_1v_2, \ldots, v_{l-1}v_l\}.$$

This path is usually denoted by $v_0v_1 \ldots v_l$. The vertices v_0 and v_l are *endvertices* of P and $l = e(P)$ is the *length* of P. We say that P is a path from v_0 to v_l or an $v_0 - v_l$ path.

A *walk* W in G is an altering sequence of vertices and edges, say $v_0, \alpha_1, v_1, \alpha_2, \ldots, \alpha_l, v_l$, where $\alpha_i = v_{i-1}v_i$, $1 \leqslant i \leqslant l$. For simplicity we write $W = v_0v_1 \ldots v_l$. Note that a path is a walk with distinct vertices. If a walk $W = v_0v_1 \ldots v_l$ is such that $l \geqslant 3$, $v_0 = v_l$, and the vertices v_i, $0 < i < l$, are distinct from each other and v_0 then W is said to be a *cycle*. The symbol P_l denotes an arbitrary path of length l and C_l denotes a cycle of length l.

Given vertices v_i, v_j, their *distance* $d(v_i, v_j)$ is the minimum length of an $v_i - v_j$ path. If there is no $v_i - v_j$ path then $d(v_i, v_j) = \infty$. A graph is *connected* if for every pair $\{v_i, v_j\}$ of distinct vertices there is a path from v_i to v_j. The *diameter* of the graph G is $\mathrm{diam}(G) = \max_{v_i, v_j} d(v_i, v_j)$. The *radius* of the graph G is $\mathrm{rad}(G) = \min_{v_i} \max_{v_j} d(v_i, v_j)$.

There are several ways to associate a matrix to a graph. The usual adjacency matrix $\boldsymbol{A}$ associated with a graph has eigenvalues quite sensitive to the maximum degree (which is a local property). The combinatorial Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$ with $\boldsymbol{D}$ denoting the diagonal degree matrix is a major tool for enumerating spanning trees and has numerous applications [32]. Another matrix associated with a graph is the (normalized) Laplacian $\tilde{\boldsymbol{L}} = \boldsymbol{I} - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$ which controls the expansion/isoperimetrical properties (which are global) and essentially determines the mixing rate of a random walk on the graph [33]. The traditional random matrices and random graphs are regular or almost regular so the spectra of all the above three matrices are basically the same (with possibly a scaling factor or a linear shift). However, for graphs with power law distribution, the above three matrices can have very different distributions [34].

Recall $\gamma_1 = 0 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_N$, repeated according to their multiplicities, are eigenvalues of the matrix $\boldsymbol{L}$. These eigenvalues are called *Laplace eigenvalues* of the graph G. Laplace eigenvalues of the complete graph K_N are $\gamma_1(K_N) = 0$ and $\gamma_k(K_N) = N$ for $2 \leqslant k \leqslant N$. The Laplace eigenvalues of the N-cycle C_N are the numbers

$$\gamma_k(C_N) = 2 - 2\cos\left(\frac{2(k-1)\pi}{N}\right), \quad k = 1, \ldots, N.$$

It is easy to see that 0 is always an eigenvalue of $\boldsymbol{L}$, and than $(1,1,\ldots,1)^{\mathrm{T}}$ is the corresponding eigenvector. More precisely, we have the following description of the multiplicity of 0 as an eigenvalue of $\boldsymbol{L}$.

Theorem 2. *The multiplicity of* 0 *as an eigenvalue of* $\boldsymbol{L}$ *is equal to the number of connected components of G.*

This implies if $\gamma_2 > 0$ then the graph is connected.

Theorem 3. *The following inequalities hold*:

$$\gamma_2(G) \leqslant \frac{N}{N-1}\delta(G) \leqslant \frac{N}{N-1}\Delta(G) \leqslant \gamma_N(G) \leqslant 2\Delta(G). \tag{19}$$

The proof of the above two theorems can be found, for example, in [36,37].

3.2. Synchronization in classical random networks

We turn now to random graphs. The primary model for the classical random graphs is the Erdös–Rényi model G_q [35], in which each edge is independently chosen with the probability q for some given $q > 0$. Let $G(N,q)$ be a random graph on N vertices.

For the model of a random graph we take a sequence of probability spaces $(\Gamma(N,q))_N$, where q is a real number between 0 and 1, and N is an integer. We shall assume that q is fixed. The probability space $\Gamma(N,q)$ consists of all labeled simple graphs on N vertices, and an edge between an arbitrary pair of vertices appears with probability q, i.e. $\Gamma(N,q)$ has 2^M elements, where $M = N(N-1)/2$, and each graph in $\Gamma(N,q)$ with m edges has the probability equal to $q^m(1-q)^{M-m}$. By $P_N(X)$ we will denote the probability of an event $X \subseteq \Gamma(N,q)$ in the probability space $\Gamma(N,q)$.

Definition 4. Almost every graph has property ρ (or ρ happens *asymptotically almost surely* (a.a.s)), if

$$\lim_{N\to\infty} P_N\{G \in \Gamma(N,q) \text{ and the graph } G \text{ has the property } \rho\} = 1.$$

Theorem 5. *Let* $G(N,q)$ *be a random graph on N vertices. Then, the class-A network* $G(N,q)$ *asymptotically almost surely synchronizes for arbitrary small coupling* σ *and the class-B network* $G(N,q)$ *asymptotically almost surely synchronizes for* $b > 1$.

Proof. The proof of the theorem follows from the following result [38]. Let q be a fixed real number between 0 and 1. For almost every graph and every $\varepsilon > 0$

$$qN - \sqrt{(2+\varepsilon)pqN\log N} < \gamma_2(G) < qN - \sqrt{(2-\varepsilon)pqN\log N} \tag{20}$$

and

$$qN + \sqrt{(2-\varepsilon)pqN\log N} < \gamma_N(G) < qN + \sqrt{(2+\varepsilon)pqN\log N}. \tag{21}$$

Therefore, for large N, $\gamma_2 \approx N$, while γ_N/γ_2 approaches 1. Now, for class-A networks the condition for synchronization reads $\sigma > a/N$ and σ can be chosen arbitrary small. For class-B networks with $b > 1$, since γ_N/γ_2 approaches 1, when $N \to \infty$, it follows that the network almost surely synchronizes. □

3.3. Synchronization in power-law networks

There are several approaches for studying power-law graphs. In the first approach, one constructs power-law graphs with prescribed degree sequence. Bender and Canfield [39] introduced a model, called configuration model, to construct a random graph with a prescribed degree sequence. This model was refined by Bollobás [37]. Recently, Molloy and Reed [40,41] used the configuration model to show that if some conditions are satisfied, then the graph almost surely has a giant component. The advantage of the configuration model is to generate graphs exactly with the prescribed degrees. However, there are several disadvantages of the configuration model. The analysis of the configuration model is much more complicated due to the dependency of the edges. A random graph from the configuration model is in fact a multi-graph instead of a simple graph. The probability of having multiple edges increases rapidly when the degrees increase.

Another line of approach is evolution models, in which one generates a vertex/edge at a time, starting from a node or a small graph. We briefly mention several such evolution models. Barabási and Albert [7] describe the following graph evolution process. Starting with a small initial graph, at each time step they add a new node and an edge between the

new node and each of m random nodes in the existing graph, where m is a parameter of the model. The random nodes are not chosen uniformly. Instead, the probability of picking a node is weighted according to its existing degree (the edges are assumed to be undirected). Using heuristic analysis with the assumption that the discrete degree distribution is differentiable, they derive a power law for the degree distribution with a power of 3, regardless of m. A power law with power 3 for the degree distribution of this model was independently derived and proved by Bollobás et al. [42]. Kumar et al. [43] proposed three evolution models: *linear growth copying*, *exponential growth copying*, and *linear growth variants*. Aiello et al. described a general random graph evolution process in [44] for generating directed power law graphs with given expected in-degrees and out-degrees. Recently, Cooper and Frieze [45] independently analyzed the above evolution of adding either new vertices or new edges and derived power law degree distribution for vertices of small degrees.

In this section we consider a random model introduced recently by Chung and Lu [46], which produces graphs with a given expected degree sequence. Therefore, this model does not produce the graph with exact given degree sequence. Instead, it yields a random graph with given expected degree sequence.

We consider the following class of random graphs with a given expected degree sequence:

$$w = (w_1, w_2, \ldots, w_N).$$

The vertex v_i is assigned a vertex weight w_i. The edges are chosen independently and randomly according to the vertex weights as follows. The probability p_{ij} that there is an edge between v_i and v_j is proportional to the product $w_i w_j$ where i and j are not required to be distinct. There are possible loops at v_i with probability proportional to w_i^2, i.e.,

$$p_{ij} = \frac{w_i w_j}{\sum_k w_k} \tag{22}$$

and we assume $\max_i w_i^2 < \sum_k w_k$. This assumption ensures that $p_{ij} \leqslant 1$ for all i and j. We denote a random graph with a given expected degree sequence w by $G(w)$. For example, a typical random graph $G(N,q)$ (see the previous section) on N vertices and edge density q is just a random graph with expected degree sequence $(qN, qN, \ldots, qN)$. The random graph $G(w)$ is different from the random graphs with an exact degree sequence such as the configuration model. We will use d_i to denote the actual degree of v_i in a random graph G in $G(w)$, where the weight w_i denotes the expected degree. The following proposition is proved in [46].

Proposition 6. *With probability* $1 - 2/N$, *all vertices* v_i *satisfy*

$$2\sqrt{w_i \log N} \leqslant d_i - w_i \leqslant \frac{2}{3}\log N + \sqrt{\left(\frac{2}{3}\log N\right)^2 + 4 w_i \log N}. \tag{23}$$

Now we give some definitions. The expected average degree d of a random graph G in $G(w)$ is defined to be

$$d = \frac{1}{N}\sum w_i. \tag{24}$$

For a subset S of vertices, the volume of S, denoted by $\mathrm{Vol}(S)$, is the sum of expected degrees in S

$$\mathrm{Vol}(S) = \sum_{v_i \in S} w_i.$$

In particular, the volume $\mathrm{Vol}(G)$ of $G(w)$ is just $\mathrm{Vol}(G) = \sum_i w_i = Nd$.

If a graph strictly follows the power law, then the average degree as well as its connectivity will be completely determined by the exponent of the power law (see [47]). However, for most realistic graphs, the power law holds only for a certain range of degrees, namely, for the degrees which are not too small and not too large. We will consider the following model [47] with the consideration that most examples of massive graphs satisfying power law have exponent $\beta > 2$.

In this paper we consider the model $M(N, \beta, d, m)$, where N is the number of vertices, $\beta > 2$ is the power of the power law, d is the expected average degree, and m is the expected maximum degree, such that $m^2 = \mathrm{o}(Nd)$ [47]. We assume that the ith vertex v_i has expected degree

$$w_i = c(i + i_0 - 1)^{-\frac{1}{\beta-1}}$$

for $1 \leqslant i \leqslant N$. Here c depends on the average degree d and i_0 depends on the maximum expected degree m. It is easy to compute that the number of vertices of expected degree between k and $k+1$ is of order $c' k^{-\beta}$, where $c' = c^{\beta-1}(\beta - 1)$, as required by the power law. To determine c, we consider

$$\mathrm{Vol}(G) = \sum_i w_i \approx c\frac{\beta-1}{\beta-2}N^{1-\frac{1}{\beta-1}}.$$

Since $Nd \approx \mathrm{Vol}(G)$, we have

$$c = \frac{\beta-2}{\beta-1}dN^{\frac{1}{\beta-1}}.$$

From

$$m = ci_0^{-\frac{1}{\beta-1}},$$

it follows:

$$i_0 = N\left[\frac{d}{m}\frac{(\beta-2)}{(\beta-1)}\right]^{\beta-1}.$$

For the considered model d can be in any range greater than 1: it does not have to grow with N [52].

Theorem 7. *Let $M(N,\beta,d,m)$ be a random power-law graph on N vertices, for which d grows with N. Assume further that d/m approaches 0 when $N \to \infty$. Then the class-A network $M(N,\beta,d,m)$ asymptotically almost surely synchronizes for arbitrary small coupling σ and class-B network $M(N,\beta,d,m)$ asymptotically almost surely does not synchronize.*

Proof. From Eq. (19),

$$\frac{N}{N-1}\Delta(M) \leqslant \gamma_N(M) \leqslant 2\Delta(M), \tag{25}$$

it follows that for large N we have $\Delta < \gamma_N \leqslant 2\Delta$, where Δ is the maximum degree of the graph. From (23) we have

$$2\sqrt{m\log N} \leqslant \Delta - m \leqslant 2\sqrt{m\log N + A^2} + 2A.$$

where $A = \log N/3$. Therefore, $\gamma_N(M) \approx \Delta$ for large N and $\gamma_N(M)$ grows with N as m.

Let k be the expected minimum degree. Then

$$k \equiv w_N = c(N+i_0-1)^{-\frac{1}{\beta-1}} \approx c(N+i_0)^{-\frac{1}{\beta-1}}.$$

Thus, we have

$$k \approx \frac{\beta-2}{\beta-1}d\left[1+\left(\frac{d}{m}\frac{(\beta-2)}{(\beta-1)}\right)^{\beta-1}\right]^{-\frac{1}{\beta-1}}. \tag{26}$$

Eq. (26) can be rewritten as

$$k \approx d\left[1+\left(\frac{d}{m}\right)^{\beta-1}\right]^{-\frac{1}{\beta-1}}.$$

Since $d/m \to 0$, when $N \to \infty$, we have $k \approx d$. Therefore, when d grows with N, the minimum expected degree k also grows with N.

It is proved in [48] that the function $\gamma_2(G)$ is non-decreasing for graphs with the same set of vertices, i.e. $\gamma_2(G_1) \leqslant \gamma_2(G_2)$ if $G_1 \subseteq G_2$ and G_1, G_2 have the same set of vertices. Let G_2 be our $M(N,\beta,d,m)$ random graph and δ be the minimum degree of the graph $M(N,\beta,d,m)$. Further, let G_1 be a δ-regular random graph which has the same set of vertices as G_2. Then obviously $G_1 \subseteq G_2$, and therefore, $\gamma_2(M) \geqslant \gamma_2(G_1)$. According to [50,49] (see also [13]), we have

$$\delta/2 - \sqrt{\delta\ln 2} \leqslant \sqrt{\gamma_2(G_1)(2\delta-\gamma_2(G_1))}.$$

Therefore,

$$\gamma_2(M) \geqslant \gamma_2(G_1) \geqslant \delta - \sqrt{\frac{3}{4}\delta^2 - \delta(\ln 2 - \sqrt{\delta\ln 2})}. \tag{27}$$

On the other hand, from Eqs. (19) and (23) it follows that for large N:

$$\gamma_2(M) \leqslant \frac{N}{N-1}\delta \approx \delta. \tag{28}$$

Combining (27) and (28) we find that $\gamma_2(M)$ can be approximated with δ. Using (23) we find that $\gamma_2(M)$ grows with N as d.

If d grows with N, since γ_2 also grows with N we conclude that the class-A network $M(N,\beta,d,m)$ almost surely synchronize for arbitrary small coupling σ. Since b is a finite number, from $\gamma_N/\gamma_2 \approx m/k \to \infty$, we see that for sufficiently large N, almost every class-B network $M(N,\beta,d,m)$ does not synchronize. □

Remark 8. Assume that d grows with N as $d = \text{const} \cdot N^{\mu}$ and $m = \text{const} \cdot N^{\nu}$. Then, clearly $\nu \geqslant \mu > 0$. Conditions of the theorem are satisfied for $\mu < \nu < (\mu+1)/2$. Indeed, from $m^2 = o(Nd)$, it follows that $\lim_{N\to\infty} m^2/(Nd) = 0$. Therefore, $\mu + 1 > 2\nu$. On the other hand, we have also assumed in the previous theorem that $\lim_{N\to\infty} d/m = 0$. Thus, we have $\nu > \mu$. Therefore, $\mu < \nu < (\mu+1)/2$. Clearly, in this case $\mu < 1$.

Remark 9. Let $M(N,\beta,d,m)$ be a random power-law graph on N vertices, for which d grows with N. Let $G(N,q)$ be a classical (Erdös–Rényi) random graph on N vertices. For both graphs γ_2 grows with N, however, as follows from the proof of this theorem, γ_2^{ER} for the classical model grows faster than $\gamma_2^{(\mathrm{pow})}$ for the power-law graph. Let σ_c be a critical value of σ for which class-A network synchronizes. Thus, the critical value $\sigma_c^{\mathrm{ER}} = a/\gamma_2^{\mathrm{ER}}$ for a classical graph is always smaller than the critical value $\sigma_c^{(\mathrm{pow})} = a/\gamma_2^{(\mathrm{pow})}$ for a power-low graph, that is $\sigma_c^{\mathrm{ER}} < \sigma_c^{(\mathrm{pow})}$.

Theorem 7 says that when $N \to \infty$ and d grows with N then class-A networks always synchronize with arbitrary small coupling, while class-B networks do not synchronize. Now we consider the case $d < \infty$. Since, in this case, we could not obtain analytical bounds for γ_2 and γ_N we provide numerical examples.

Consider the model $M(N,\beta,d,m)$ with $\beta = 3$, $d = 7$, and $m = 30$. Figs. 1–3 show the γ_2, γ_N, and γ_N/γ_2 versus N. The figures are obtained by simulating graphs composed of 200–1200 nodes, with a step of 10 nodes. For each case, 10 different simulations are computed and the mean value is presented as a dot (solid line is a curve fitting the dots). Note that the actual maximum degree Δ may differ from the expected maximum degree m. Consider now a class-A network with $a = 1$ and a class-B network with $b = 40$. From Fig. 1 one can compute the value of γ_2 for $N = 1200$, $\gamma_2 = 0.31$, and

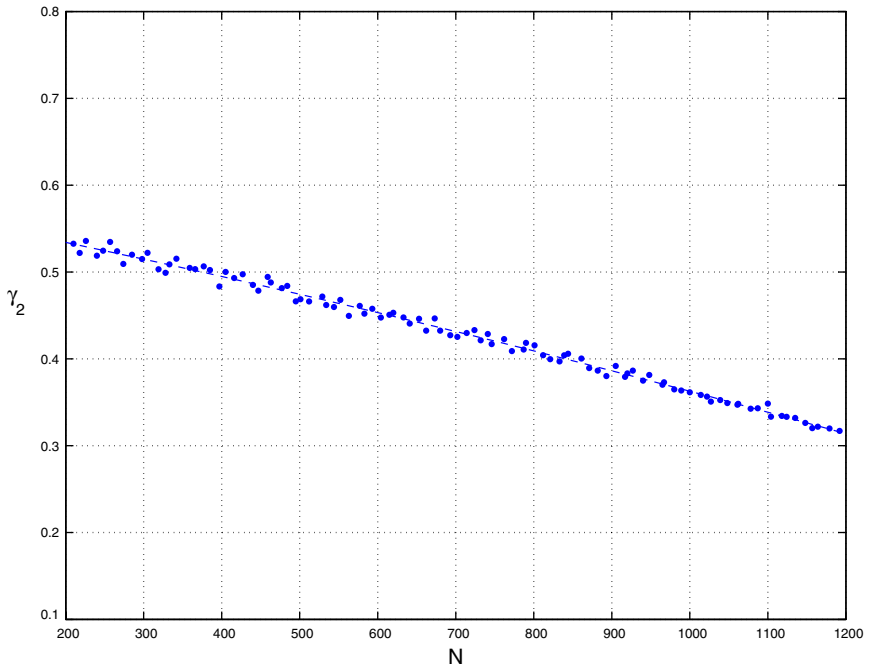

Fig. 1. γ_2 versus N for the model $M(N,\beta,d,m)$ with $\beta = 3$, $d = 7$, and $m = 30$.

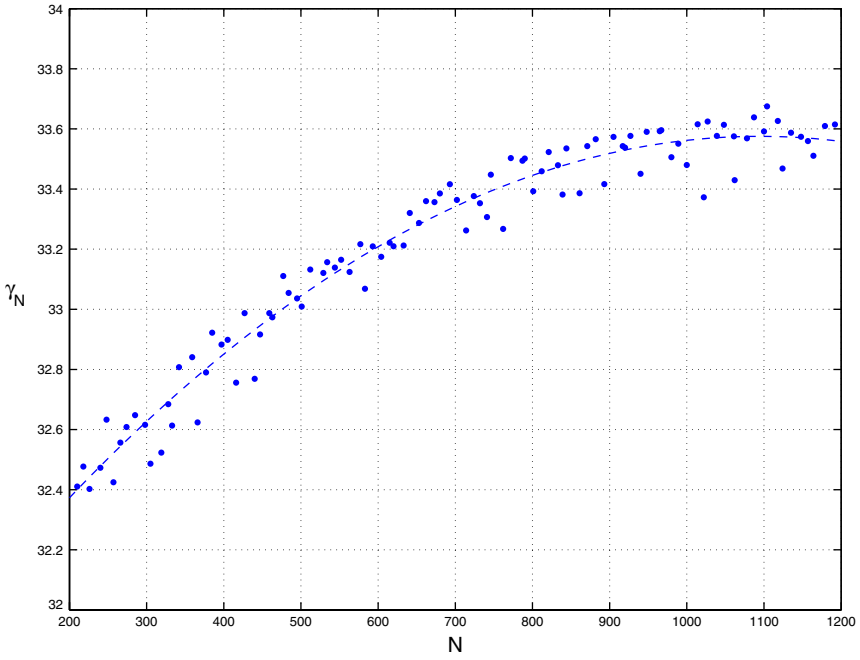

Fig. 2. γ_N versus N for the model $M(N,\beta,d,m)$ with $\beta = 3$, $d = 7$, and $m = 30$.

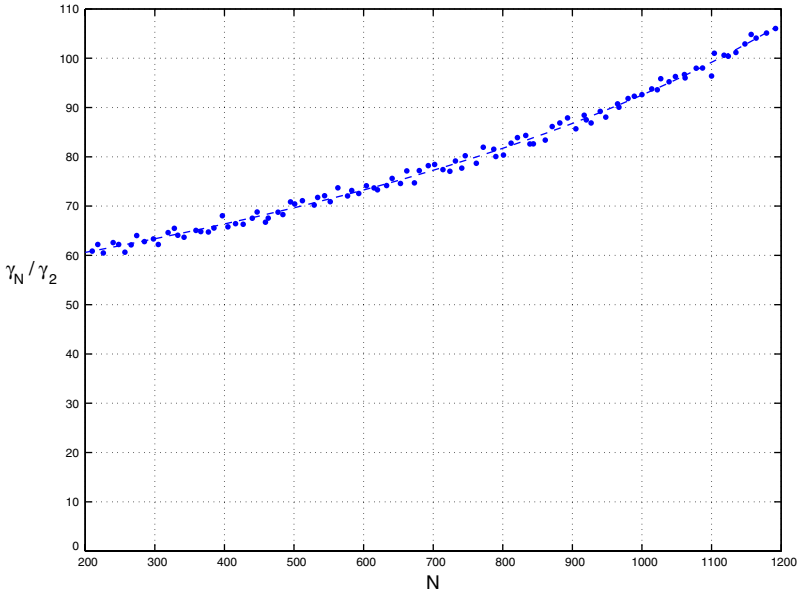

Fig. 3. γ_N/γ_2 versus N for the model $M(N,\beta,d,m)$ with $\beta = 3$, $d = 7$, and $m = 30$.

therefore, the network synchronizes for $\sigma > 3.23$. Moreover, from Fig. 3 one can compute the value of γ_N/γ_2 for $N = 1200$, which is approximately $\gamma_N/\gamma_2 = 107$. Consequently, since $b < 107$, the class-B network does not synchronizes.

Let us write $\sigma_c = a/\gamma_2$ and $b_c = \gamma_N/\gamma_2$. σ_c and b_c are critical values for which the network may synchronize, in other words, if $\sigma > \sigma_c$ ($b > b_c$), then the class-A (class-B) network synchronizes. The proof of Theorem 7 suggests that the critical values may be approximated as $\sigma_c \approx a/k$ and $b_c = m/k$ provided that k and δ are close to each other. For example, consider a network composed by $N = 1200$ nodes with $d = 20$, $m = 200$, and $\beta = 3$, for which $k \simeq 9.99$. Then we

have (with $a = 1$ and $b = 40$) $\sigma_c \simeq 0.10$ and $b_c \simeq 20.02$. Simulating such a network, the following actual eigenvalues have been obtained: $\gamma_2^{(act)} \simeq 7.61$, $\gamma_N^{(act)} \simeq 196.43$, and $(\gamma_N/\gamma_2)^{(act)} \simeq 25.83$. It follows that the actual critical values are $\sigma_c^{(act)} \simeq 0.13$ and $b_c^{(act)} \simeq 25.83$. In this case, since $b = 40$, both class-A and class-B networks synchronize.

4. Synchronization in hybrid networks

It has been observed that many realistic networks possess the so-called small-world phenomenon, with two distinguishing properties: small distance between any pair of nodes, and the clustering effect that two nodes are more likely to be adjacent if they share a neighbor. In this section, we consider a hybrid graph model proposed by Chung and Lu [51], which has both aspects of the small-world phenomenon. Roughly speaking, a hybrid graph is a union of a global graph (consisting of "long edges" providing small distances) and a local graph (consisting of "short edges" respecting local connections).

Examples of local graphs include paths and cycles. More generally, we define a local graph as follows: consider a lattice graph where the vertices are in a d-dimensional lattice where each vertex is a d-dimensional vector in the hypercube $[0, r]^d$ with integer entries. Suppose each vertex is connected to its nearest neighbors. This graph, also known as the grid graph, has diameter D, which is a function of the number of vertices N, and has maximum vertex degree $\Delta = 2d$.

Theorem 10. *When $N \to \infty$ local graphs for both class-A and class-B oscillators do not synchronize.*

Proof. It is know that, see for example [13],

$$\gamma_2 < \frac{2d \ln(N-1)}{2(D-2) - \ln(N-1)}$$

if $2(D-2) - \ln(N-1) > 0$. Therefore, $\gamma_2 \to 0$ as $N \to \infty$ for the grid graphs. This is also true when the vertices are connected to neighbors in an arbitrary local neighborhood. On the other hand, $2d = \Delta(G) \leqslant \gamma_N \leqslant 2\Delta(G) = 4d$. Therefore, $\gamma_N/\gamma_2 \to \infty$ as $N \to \infty$. □

A hybrid graph consists of two parts: a global graph and a local graph. The edge set of the hybrid graph is a disjoint union of the edge set of the global graph G and that of the local graph L. We consider two cases: classical random graph model $G(N, q)$, described in Section 3.2, and power-law random graph model $M(N, \beta, d, m)$, described in Section 3.3. For local graph L we consider the grid graph, although other choices are also possible. For example, Chung and Lu use two parameters to describe the local connectivity. For any fixed two integers $k \geqslant 2$ and $l \geqslant 2$, a graph L is called *locally (k, l)-connected* if for any edge $v_i v_j$, there are at least l edge-disjoint paths with length at most k joining from v_i to v_j (including the edge $v_i v_j$). For any two points v_i and v_j, the probability of choosing an edge between v_i and v_j is denoted by $p(v_i, v_j)$, defined as follows:

- $p(v_i, v_j) = 1$ if $v_i v_j$ is an edge of L;
- $p(v_i, v_j) = q$ for a classical random graph;
- $p(v_i, v_j) = \rho w_i w_j$ for a power-law graph.

Let now consider a hybrid network for which equation of the motion can be written as

$$\dot{\boldsymbol{x}} = \mathbf{F}(\boldsymbol{x}) + \sigma[(\boldsymbol{L}_L + \boldsymbol{L}_G) \otimes \boldsymbol{H}]\boldsymbol{x}, \tag{29}$$

where $\boldsymbol{L}_L$ is the matrix describing the local graph L, and $\boldsymbol{L}_G$ is the coupling matrix of the global graph G. Let $N_{\text{total}} = N(N-1)/2$ be the total number of edges (links) in a network with N nodes and N_L be the total number of local edges. Then $N_G = N_{\text{total}} - N_L$ is the number of all possible global edges. Let pN_G, where $0 \leqslant p \leqslant 1$, be a number of global edges.

Theorem 11. *Assume N is large enough and let G be a global graph (classical random graph model or power-law model). Then for class-A networks, given a, there exist a number p, such that $\sigma_c(p) \ll \sigma_c(0)$, where $\sigma_c(0) = a/\gamma_2(0)$, $\sigma_c(p) = a/\gamma_2(p)$, $\gamma_2(p)$ is the second eigenvalue of the matrix $\boldsymbol{L}_L + \boldsymbol{L}_G$, and $\gamma_2(0)$ is the second eigenvalue of the matrix $\boldsymbol{L}_L$. For class-B networks, given $b > 1$, there exist a number p, such that $\gamma_N(p)/\gamma_2(p) < b$, where $\gamma_2(p)$ and $\gamma_N(p)$ are the second and the Nth eigenvalue, respectively, of the matrix $\boldsymbol{L}_L + \boldsymbol{L}_G$.*

Proof. Since for $p = 1$, the matrix $\boldsymbol{L}_L + \boldsymbol{L}_G$ is fully connected, it follows that $\gamma_i(1) = N$, $i \geqslant 2$; hence $\gamma_2(1) = N$ and $\gamma_N(1)/\gamma_2(1) = 1$. On the other hand, on average, the $\gamma_2(p)$ is monotonically increasing function of p and $\gamma_N(p)/\gamma_2(p)$ is monotonically decreasing function of p. Thus, for both classes of networks (class-A and class-B), there exists a critical value of p, p_c, such that for $p > p_c$, almost all networks (29) are synchronizable. □

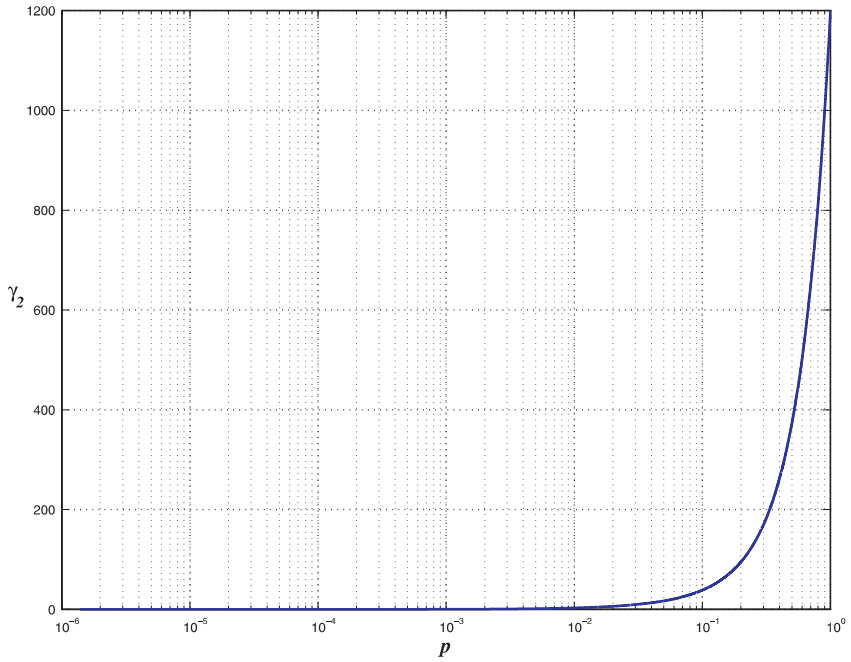

Fig. 4. γ_2 versus p for the hybrid model with $N = 1200$, in which the local graph is a circle and the global graph is a classical random graph model.

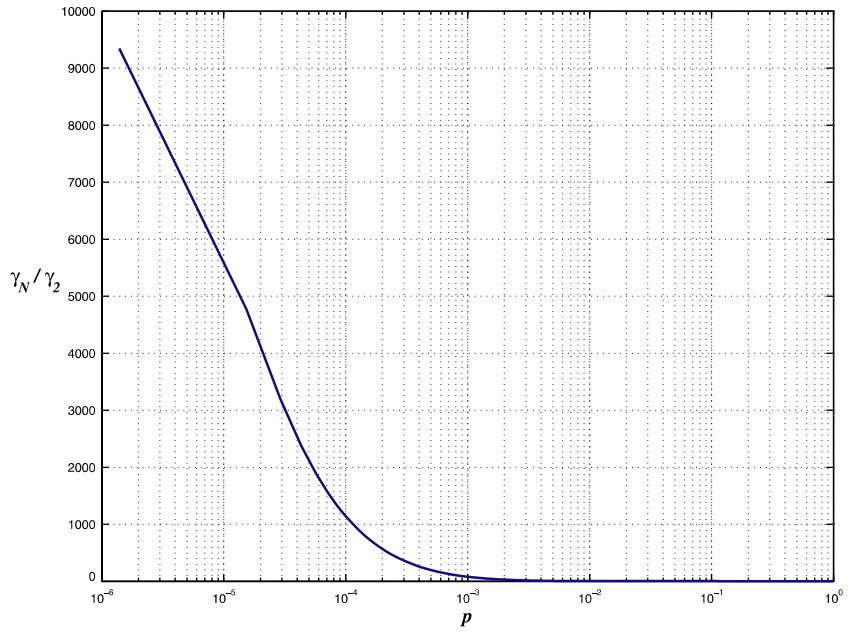

Fig. 5. γ_N/γ_2 versus p for the hybrid model with $N = 1200$, in which the local graph is a circle and the global graph is a classical random graph model.

We now present an example. Let the local graph L be a circle and $N = 1200$. It is easy to compute that $\gamma_2(0) = 8.3513 \times 10^{-9}$ and $\gamma_N(0)/\gamma_2(0) = 1436156.321$.

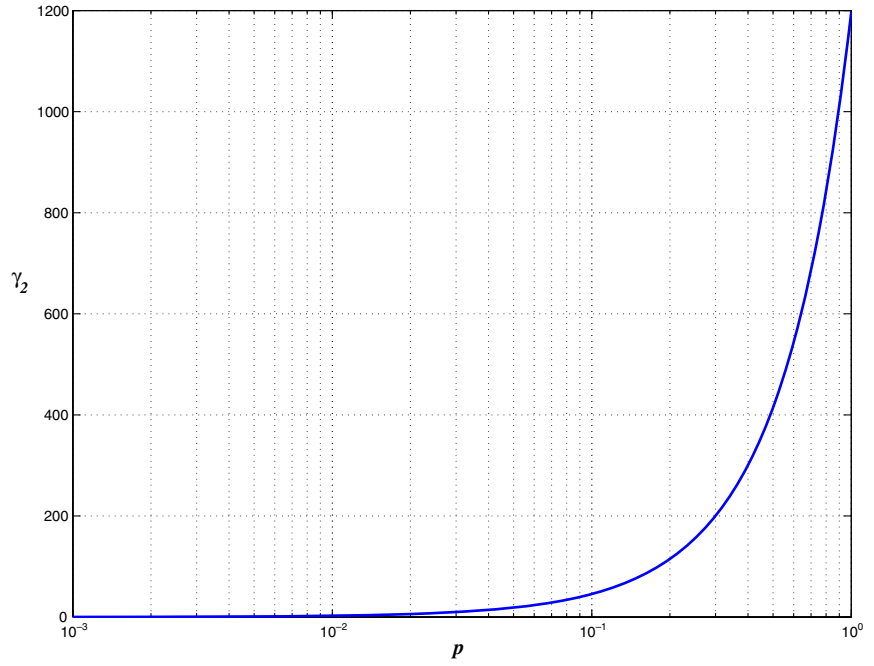

Fig. 6. γ_2 versus p for the hybrid model with $N = 1200$ and $m = 5$, in which the local graph is a circle and the global graph is a power-law graph model.

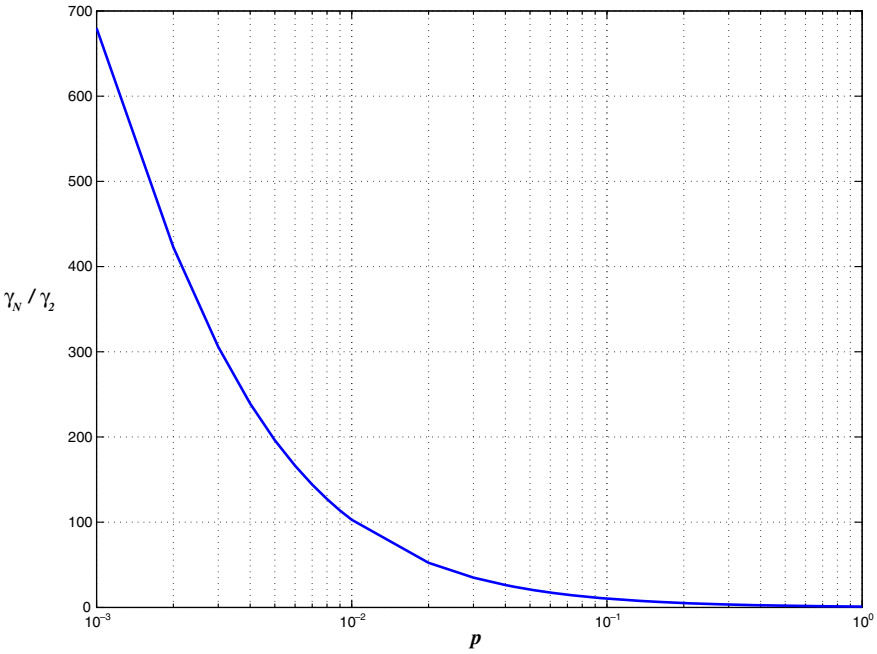

Fig. 7. γ_N/γ_2 versus p for the hybrid model with $N = 1200$ and $m = 5$, in which the local graph is a circle and the global graph is a power-law graph model.

Assume that the global graph is a classical random graph model. Consider first class-A oscillators for which $a = 1$ and $\sigma \leqslant 10$. Since $\sigma\gamma_2 \ll 1$, the local network L of 1200 oscillators does not synchronize. The dependence of $\gamma_2(p)$ on p is shown in Fig. 4. Since $\sigma\gamma_2 > a$, it follows that the hybrid graph $L + G$ synchronizes if $\gamma_2(p) > a/\sigma = 0.1$. From Fig. 4 one easily finds that $\gamma_2(p) > 0.1$ already for $p = 33.30 \times 10^{-4}$. We consider now a network of class-B oscillators for which $b = 40$. Since $\gamma_N/\gamma_2 \gg 40$, the local network L does not synchronize. The dependence of $\gamma_N(p)/\gamma_2(p)$ on p is shown in Fig. 5. Since the condition for synchronization is $\gamma_N/\gamma_2 < b$, it follows that the hybrid graph $L + G$ synchronizes for $p = 15.78 \times 10^{-4}$. Therefore, adding only a small number of global edges makes the oscillators synchronize.

Assume now that the global graph is a power-law graph model. Numerically we consider the model generated in the following way. First, we choose m nodes at random from all N nodes with equal probabilities and assign them to be centers. Second, we add links (global edges) by connecting one node chosen at random from all N nodes to another node randomly chosen from the m centers. Third, when all centers are fully connected with other nodes, we start uniformly to add links between the rest of the nodes. The dependence of $\gamma_2(p)$ and $\gamma_N(p)/\gamma_2(p)$ on p for such model is shown in Figs. 6 and 7, respectively for $m = 5$. From these figures and our numerical experiments, we may conclude: (i) $\gamma_N(p)$ increases reaching the maximum values N for smaller value of m; thus, γ_N reaches the value N in the fastest way for $m = 1$, and (ii) γ_2 is not effected by m. Therefore, the random model with m centers only influences synchronization property of class-B networks: if one adds global edges using the model with centers, the network becomes more difficult to synchronize. Thus, for example, class-B network with $b = 40$ will synchronize for $p = 26.70 \times 10^{-3} > 15.78 \times 10^{-4}$. Saying in another way, if the global edges are added independently, then the synchronization is optimal.

5. Conclusion

In this paper we studied synchronization in networks with different topologies. We can summarize the main conclusions of this paper as follows:

- Let $G(N, q)$ be a classic random graph (Erdös–Rényi model) on N vertices. We proved that for sufficiently large N, the class-A network $G(N, q)$ almost surely synchronize for arbitrary small coupling σ. For sufficiently large N, almost every class-B network $G(N, q)$ with $b > 1$ synchronizes.
- Let $M(N, \beta, d, m)$ be a random power-law graph on N vertices. We proved that for sufficiently large N, the class-A network $M(N, \beta, d, m)$ almost surely synchronize for arbitrary small coupling σ. For sufficiently large N, almost every class-B network $M(N, \beta, d, m)$ does not synchronize.
- Let $M(N, \beta, d, m)$ be a random power-law graph on N vertices, for which d grows with N. Let $G(N, q)$ be a classical (Erdös–Rényi) random graph on N vertices. Let σ_c^{ER} and $\sigma_c^{(pow)}$ be critical values of σ for which class-A classical random network synchronizes and class-A power-law random network synchronizes, respectively. Then $\sigma_c^{ER} < \sigma_c^{(pow)}$.
- Adding small number of global edges to a local graph makes the corresponding hybrid graph to synchronize.

References

[1] Barabási AL. Linked: the new science of networks. Cambridge, Massachusetts: Perseus Publishing; 2002.
[2] Cohen JE, Briand F, Newman CM. Community food webs: data and theory. Berlin: Springer; 1990.
[3] Williams RJ, Martinez ND. Simple rules yield complex food webs. Nature 2000;404:180–3.
[4] Broder A, Kumar R, Maghoul F, Raghavan P, Rajagopalan S, Stata R, et al. Graph structure in the web. Comput Networks 2000;33:309–20.
[5] Faloutsos M, Faloutsos P, Faloutsos C. On power-law relationships of the Internet topology. Comput Commun Rev 1999;29(4):251–62.
[6] Watts DJ, Strogatz SH. Collective dynamics of 'small-world' networks. Nature 1998;393:440–2.
[7] Barabási AL, Albert R. Emergence of scaling in random networks. Science 1999;286:509–12.
[8] Albert R, Barabási AL. Statistical mechanics of complex networks. Rev Mod Phys 2002;74:47–97.
[9] Fujisaka H, Yamada T. Stability theory of synchronized motion in coupled-oscillator systems. Progr Theor Phys 1983;69(1):32–47.
[10] Strogatz S. Sync: the emerging science of spontaneous order. New York: Hyperion; 2003.
[11] Wu CW, Chua LO. Synchronization in an array of linearly coupled dynamical systems. IEEE Trans Circ Syst I 1995;42:430–47.
[12] Wu CW. Synchronization in arrays of coupled chaotic circuits and systems: theory and applications. In: Chen G, editor. Controlling chaos and bifurcations in engineering systems. Boca Raton (FL): CRC Press; 1999 [chapter 23].

[13] Wu CW. Synchronization in arrays of coupled nonlinear systems: passivity, circle criterion, and observer design. IEEE Trans Circ Syst I 2001;48:1257–61.
[14] Wu CW. Synchronization in coupled chaotic circuits and systems. Singapore: World Scientific; 2002.
[15] Wu CW. Synchronization in coupled arrays of chaotic oscillators with nonreciprocal coupling. IEEE Trans Circ Syst I 2003;50:294–7.
[16] Wu CW. Perturbation of coupling matrices and its effect on the synchronizability in arrays of coupled chaotic systems. Phys Lett A 2003;319:495–503.
[17] Pecora L, Carroll T. Master stability functions for synchronized coupled systems. Phys Rev Lett 1998;80(10):2109–12.
[18] Fink KS, Johnson G, Carroll T, Mar D, Pecora L. Three coupled oscillators as a universal probe of synchronization stability in coupled oscillator arrays. Phys Rev E 2000;61(5):5080–90.
[19] Barahona M, Pecora LM. Synchronization in small-world systems. Phys Rev Lett 2002;89(5). 054101(1-4).
[20] Ott E. Chaos in dynamical systems. Cambridge, UK: Cambridge University Press; 1993.
[21] Wang XF, Chen G. Synchronization in scale-free dynamical networks: robustness and fragility. IEEE Trans Circ Syst I 2002;49(1):54–62.
[22] Nishikawa T, Motter AE, Lai Y-C, Hoppensteadt FC. Heterogeneity in oscillator networks: are smaller worlds easier to synchronize? Phys Rev Lett 2003;91:014101.
[23] Li X, Chen G. Synchronization and desynchronization of complex dynamical networks: an engineering viewpoint. IEEE Trans Circ Syst I 2003;50(11):1381–90.
[24] Lu J, Yu X, Chen G, Cheng D. Characterizing the synchronizability of small-world dynamical networks. IEEE Trans Circ Syst I: Regular papers 2004;51(4):787–96.
[25] Zhou C, Motter AE, Kurths J. Universality in the synchronization of weighted random networks. Phys Rev Lett 2006;96:034101.
[26] Belykh VN, Belykh IV, Hasler M. Connection graph stability method for synchronized coupled chaotic systems. Physica D 2004;195:159–87.
[27] Belykh IV, Belykh VN, Hasler M. Blinking model and synchronization in small-world networks with a time-varying coupling. Physica D 2004;195:188–206.
[28] Rugh WJ. Linear system theory. 2nd ed. Upper Saddle River (NJ), USA: Prentice Hall; 1996.
[29] Checco P, Kocarev L, Maggio GM, Biey M. On the synchronization region in networks of coupled oscillators. ISCAS 2004;IV:800–3.
[30] Stojanovski T, Kocarev L, Parlitz U, Harris R. Sporadic driving of dynamical systems. Phys Rev E 1997;55:4035–48.
[31] Madan RN. Chua's circuit: a paradigm for chaos. Singapore: World Scientific; 1993.
[32] Biggs N. Algebraic graph theory. Cambridge University Press; 1993.
[33] Chung F. Spectral graph theory. AMS Publications; 1997.
[34] Chung F, Lu L, Vu V. The spectra of random graphs with given expected degrees. Proc Natl Acad Sci 2003;100(11):6313–8.
[35] Erdös P, Rényi A. On random graphs. Publ Math Debrecen 1959;6:290–1.
[36] Bollobás B. Graph theory: an introductory course. New York: Springer-Verlag; 1979.
[37] Bollobás B. Random graphs. 2nd ed. Cambridge: Cambridge University Press; 2001.
[38] Juvan M, Mohar B. Laplace eigenvalues and bandwidth-type invariants of graphs. J. Graph Theory 1993;17:393–407.
[39] Bender EA, Canfield ER. The asymptotic number of labelled graphs with given degree sequences. J Combinat Theory (A) 1978;24:296–307.
[40] Molloy M, Reed B. A critical point for random graphs with a given degree sequence. Rand Struct Algorith 1995;6(2–3):161–79.
[41] Molloy M, Reed B. The size of the giant component of a random graph with a given degree sequence. Combinat Probab Comput 1998;7(3):295–305.
[42] Bollobás B, Riordan O, Spencer J, Tusnady G. The degree sequence of a scale-free random graph process. Rand Struct Algorith 2001;18(3):279–90.
[43] Kumar R, Raghavan P, Rajagopalan S, Sivakumar D, Tomkins A, Upfal E, Stochastic models for the web graph. In: Proceedings of the 41st annual symposium on foundations of computer science, 2000. p. 57–65.
[44] Aiello W, Chung F, Lu L. Random evolution in massive graphs. In: Abello J et al., editors. Handbook on massive data sets. Kluwer Academic Publishers; 2002. p. 97–122.
[45] Cooper C, Frieze A. A general model of web graphs. In: Proceedings of ESA 2001, 2001. p. 500–11.
[46] Chung F, Lu L. Connected components in random graphs with given expected degree sequences. Ann Combinat 2001;6:125–45.
[47] Aiello W, Chung F, Lu L. A random graph model for massive graphs. In: Proceedings of the thirty-second annual ACM symposium on theory of computing, 2000. p. 171–80.
[48] Fiedler M. Algebraic connectivity of graphs. Czech Math J 1973;23(98):298–305.
[49] Mohar B. Isoperimetric numbers of graphs. J Combinat Theo Ser B 1989;47:274–91.
[50] Bollobás B. The isoperimetric number of random regular graphs. Euro J Combinat 1988;9:241–4.
[51] Chung F, Lu L. The small world phenomenon in hybrid power law graphs. In: Ben-Naim E, Frauenfelder H, Toroczkai Z, editors. Complex networks. Springer-Verlag; 2004. p. 91–106.
[52] F. Chung, personal communication.