Vector field, phase plane, phase portrait

1.2.1 Trajectories and Equilibria (Seydel, Practical Bifurcation and Stability Analysis, p.6-7)

Many kinds of qualitative changes can be described by systems of ordinary differential equations (ODEs). In this section some elementary facts are recalled and notation is introduced; compare also Appendix A.3.

Suppose the state of the system is described by functions $x_1(t)$ and $x_2(t)$. These *state variables* may represent, for example, the position and velocity of a particle, concentrations of two substances, or electric potentials. The independent variable t is often "time." Let the physical (or chemical, etc.) law that governs $x_1(t)$ and $x_2(t)$ be represented by two ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2). \end{aligned} \qquad (1.1)$$

In vector notation equation (1.1) can be written as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. This system is called *autonomous*, because the functions f_1 and f_2 do not depend explicitly on the independent variable t. As illustrated in Figure 1.4, solutions $x_1(t)$ and $x_2(t)$ of this *two-dimensiona.l* system form a set of *trajectories (flow lines, orbits)* densely covering part or all of the (x_1, x_2) plane. A figure like Figure 1.4 depicting selected trajectories is called a *phase diagram;* the (x_1, x_2) plane is the *phase plane*. A specific trajectory is selected by requesting that it pass through a prescribed point (z_1,z_2) of *initial values*

$$x_1(t_0) = z_1, \quad x_2(t_0) = z_2$$

Because the system is autonomous, we can take $t_0 = 0$. Imagine a tiny particle floating over the (x_1, x_2) plane, the tangent of its path being given by the differential equations. After time t_1 has elapsed, the particle will be at the point

$$(x_1(t_1), x_2(t_1))$$

(see Figure 1.4). The set of all trajectories that start from some part of the $\mathbf{x}$ plane is called the *flow* from that part.

Of special interest are *equilibrium* points (x_1^s, x_2^s) which are defined by

$$\dot{x}_1 = 0, \quad \dot{x}_2 = 0$$

In equilibrium points, the system is at rest; equilibrium solutions are constant solutions.

These points (x_1^s, x_2^s) are also called *stationary* solutions, and seldom singular points, critical points, rest points, or fixed points. Stationary points are solutions of the system of equations given by the *right-hand side* $\mathbf{f}$ of the differential equation,

$$\begin{aligned} f_1(x_1^s, x_2^s) &= 0 \\ f_2(x_1^s, x_2^s) &= 0 \end{aligned}$$

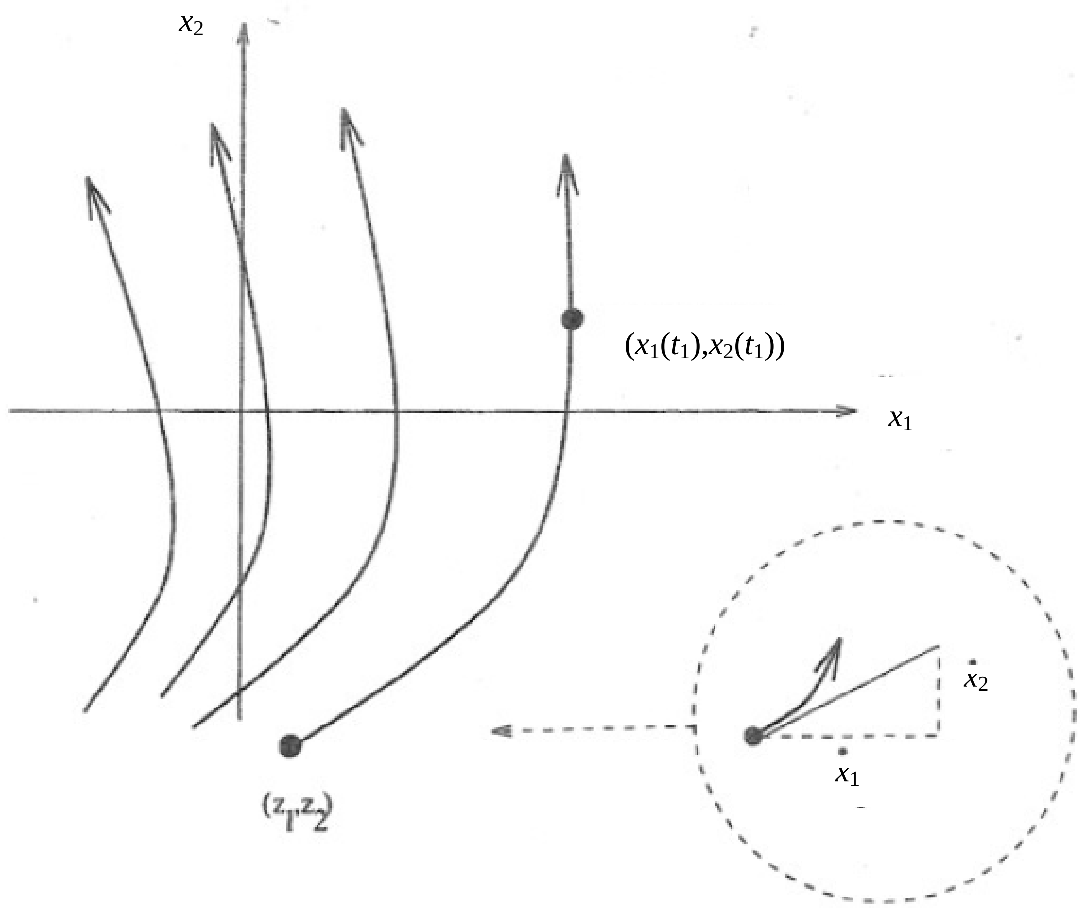

FIGURE 1.4. Trajectories in a phase plane.

In vector notation, this system of equations is written as $\mathbf{f}(\mathbf{x}^s) = \mathbf{0}$, which defines stationary solutions $\mathbf{x}^s$. For a practical evaluation of stationary points in two dimensions, we observe that each of these two equations defines a curve named *null cline*. Hence, stationary solutions are intersections of null clines. This suggests starting with drawing the null clines in a phase plane. In this way, not only the equilibria are obtained, but also some information on the global behavior of the trajectories. Recall that trajectories intersect the null cline defined by $f_1(x_1, x_2) = 0$ vertically and intersect the null cline of $f_2(x_1, x_2) = 0$ horizontally.

(Khalil, p. 24 -28)

1.2 Second-Order Systems

Second-order autonomous systems occupy an important place in the study of nonlinear systems because solution trajectories can be represented by curves in the plane. This allows easy visualization of the qualitative behavior of the system. The purpose of this section is to use second-order systems to introduce, in an elementary context, some of the basic ideas of nonlinear systems. In particular, we shall look at the behavior of a nonlinear system near equilibrium points and the phenomenon of nonlinear oscillation.

A second-order autonomous system is represented by two scalar differential equations:

$$\dot{x}_1 = f_1(x_1, x_2) \tag{1.5}$$

$$\dot{x}_2 = f_2(x_1, x_2) \tag{1.6}$$

where $f_1(\cdot,\cdot)$ and $f_2(\cdot,\cdot)$ are maps from R^2 into R. Let $x(t) = (x_1(t), x_2(t))$ be the solution[4] of (1.5)–(1.6) that starts at a certain initial state $x_0 = (x_{10}, x_{20})$; that is, $x(0) = x_0$. The solution $x(t)$ is a map from the real numbers R into R^2. The locus in the x_1–x_2 plane of the solution $x(t)$ for

[4]It is assumed that there is a unique solution.

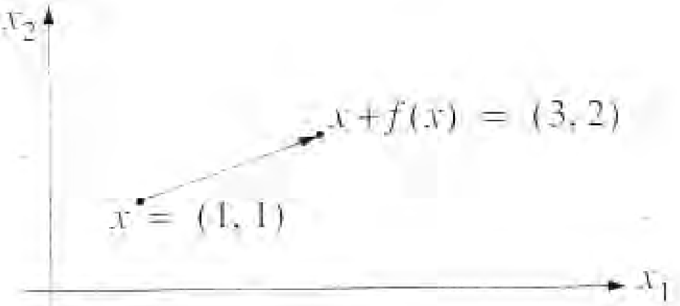

Figure 1.10: Vector field representation.

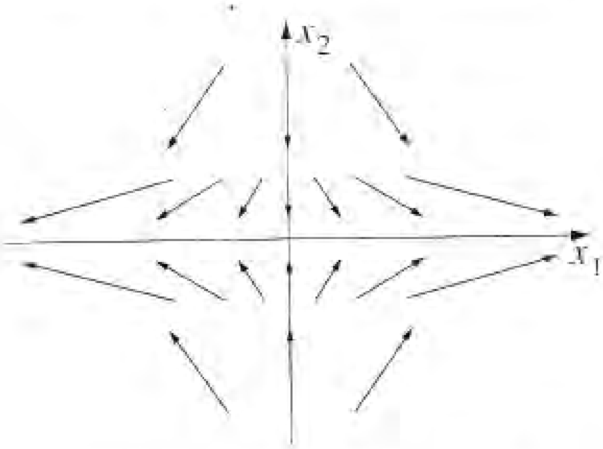

Figure 1.11: Vector field diagram.

all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit* of (1.5)–(1.6) from x_0. The x_1–x_2 plane is usually called the *state plane* or *phase plane*. The right-hand side of (1.5)–(1.6) expresses the tangent vector $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t))$ to the curve. Using the vector notation

$$\dot{x} = f(x)$$

where $f(x)$ is the vector $(f_1(x), f_2(x))$, we consider the map $f(\cdot)$ as a *vector field* on the plane R^2. This means that to each point x in the plane we assign a vector $f(x)$. For easy visualization, we represent $f(x)$ as a vector based at x; that is, we assign to x the directed line segment from x to $x + f(x)$. For example, if $f(x) = (2x_1^2, x_2)$, then at $x = (1, 1)$ we draw an arrow pointing from $(1, 1)$ to $(1, 1) + (2, 1) = (3, 2)$; see Figure 1.10. Repeating this at every point in the plane, we obtain the vector field diagram in Figure 1.11. In this figure, the length of the arrow at a given point x is proportional to the

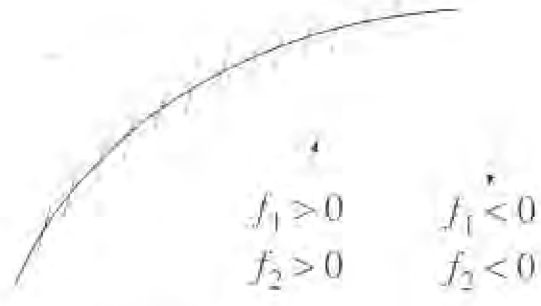

Figure 1.12: Isocline with positive slope.

Euclidean norm of $f(x)$. Since the vector field at a point is tangent to the trajectory through that point, we can, in essence, construct the trajectory starting at a given point x_0 from the vector field diagram.

The family of all trajectories or solution curves is called the *phase portrait* of (1.5)–(1.6). An (approximate) picture of the phase portrait can be constructed by plotting trajectories from a large number of initial states spread (usually uniformly) all over the x_1–x_2 plane. Since numerical subroutines for solving general nonlinear state equations are widely available, we can easily construct the phase portrait using computer simulations. There are, however, several other methods for constructing the phase portrait without using computers. While we do not need to dwell on describing these methods, it might be useful to know one of them. The method we chose to describe here is known as the *isocline method*. To understand the idea behind the method, note that the slope of a trajectory at any given point x, denoted by $s(x)$, is given by

$$s(x) = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

Therefore, the equation

$$s(x) = c$$

defines a curve in the x_1–x_2 plane along which the trajectories of (1.5)–(1.6) have slope c. Thus, whenever a trajectory of (1.5)–(1.6) crosses the curve $s(x) = c$, the slope of the trajectory at the intersection point must be c. The procedure is to plot the curve $s(x) = c$ in the x_1–x_2 plane and along this curve draw short line segments having the slope c. These line segments are parallel and their directions are determined by the signs of $f_1(x)$ and $f_2(x)$ at x; see Figure 1.12. The curve $s(x) = c$ is known as an isocline. The procedure is repeated for sufficiently many values of the constant c, until

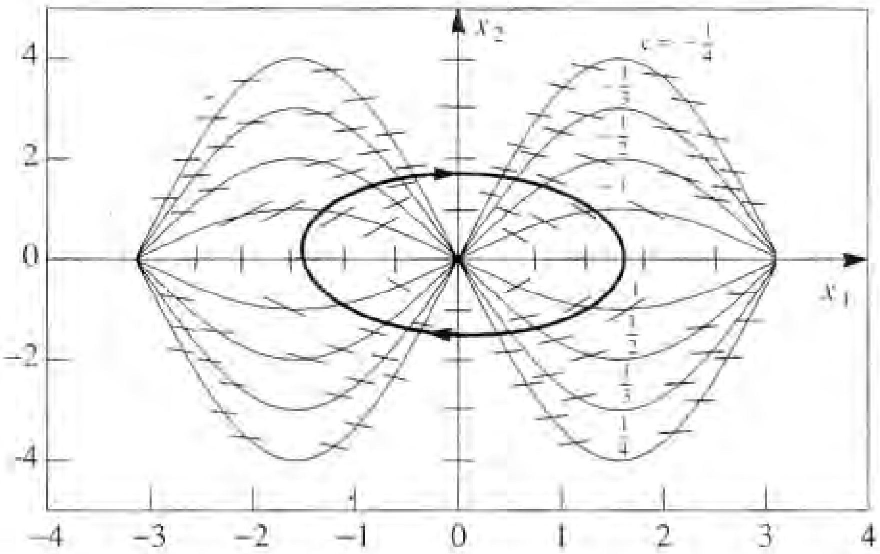

Figure 1.13: Graphical construction of the phase portrait of the pendulum equation (without friction) by the isocline method.

the plane is filled with isoclines. Then, starting from a given initial point x_0, one can construct the trajectory from x_0 by moving, in the direction of the short line segments, from one isocline to the next.

Example 1.1 Consider the pendulum equation without friction:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sin x_1\end{aligned}$$

The slope function $s(x)$ is given by

$$s(x) = \frac{-\sin x_1}{x_2}$$

Hence, the isoclines are defined by

$$x_2 = -\frac{1}{c}\sin x_1$$

Figure 1.13 shows the isoclines for several values of c. One can easily sketch the trajectory starting at any point, as shown in the figure for the trajectory starting at $(\pi/2, 0)$. If the sketch is done carefully, one might even detect that the trajectory is a closed curve. Consider next the pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -0.5x_2 - \sin x_1\end{aligned}$$

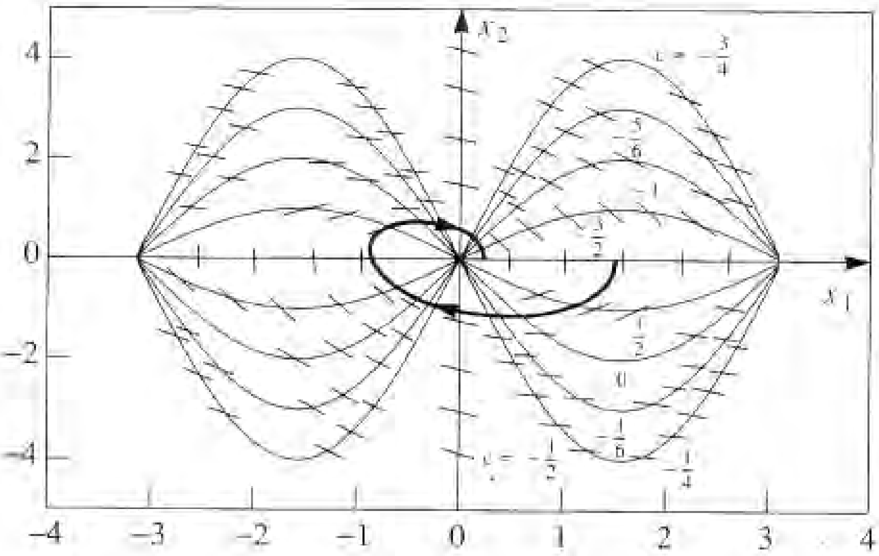

Figure 1.14: Graphical construction of the phase portrait of the pendulum equation (with friction) by the isocline method.

This time the isoclines are defined by

$$x_2 = -\frac{1}{0.5 + c}\sin x_1$$

The isoclines shown in Figure 1.14 are identical to those of Figure 1.13, except for the slopes of the line segments. A sketch of the trajectory starting at $(\pi/2, 0)$ shows that the trajectory is a shrinking spiral that moves toward the origin. △

Note that since the time t is suppressed in a trajectory, it is not possible to recover the solution $(x_1(t), x_2(t))$ associated with a given trajectory. Hence, a trajectory gives only the *qualitative* but not *quantitative* behavior of the associated solution. For example, a closed trajectory shows that there is a periodic solution; that is, the system has a sustained oscillation, whereas a shrinking spiral shows a decaying oscillation. In the rest of this section we will analyze qualitatively the behavior of second-order systems using their phase portraits.