Qualitative behaviour of nonlinear systems

(from Strogatz, Nonlinear dynamics and chaos, Perseus Books, Cambridge, Ma, 1994, Ch. 6)

Let us consider a simple second-order autonomous nonlinear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \tag{1}$$

Equation (1) is equivalent to

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2) \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2) \end{cases} \tag{2}$$

First, we will classify the kinds of fixed points that can arise, using our knowledge of linear systems, then we will try to determine the *qualitative* behaviour of the solutions, by finding the qualitative system phase portrait directly from the properties of $\mathbf{f}(\mathbf{x})$.

Linearization around a fixed point

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

$\underline{x}^* = (x_1^*, x_2^*)$: fixed point

i.e. $f_1(x_1^*, x_2^*) = 0$

$f_2(x_1^*, x_2^*) = 0$

Poniamo $x_1 = x_1^* + \delta_1$ $\quad x_2 = x_2^* + \delta_2$

$\dot{\delta}_1 = \dot{x}_1$ (poiché x_1^* è una costante)

Quindi:

$$\dot{\delta}_1 = f_1(x_1^* + \delta_1, x_2^* + \delta_2) = f_1(x_1^*, x_2^*) + \left.\frac{\partial f_1}{\partial x_1}\right|_{\underline{x}^*} \delta_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{\underline{x}^*} \delta_2 + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)$$

$$\simeq \frac{\partial f_1}{\partial x_1}\delta_1 + \frac{\partial f_1}{\partial x_2}\delta_2$$

(derivate valutate in $\underline{x}^*$)

In modo simile

$$\dot{\delta}_2 \simeq \frac{\partial f_2}{\partial x_1}\delta_1 + \frac{\partial f_2}{\partial x_2}\delta_2$$

Quindi :

$$\begin{bmatrix} \dot{\delta}_1 \\ \dot{\delta}_2 \end{bmatrix} \cong \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

Se il sistema linearizzato predice una sella, un nodo o un fuoco, allora tale è pure il punto di equilibrio del sistema non lineare.

The *exact* nonlinear state equation (4.2) and the *linearized* state equation (5.24) about the equilibrium point $\mathbf{x}_Q$ are written in vector form as follows for comparison purposes:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \text{or} \qquad \dot{\tilde{\mathbf{x}}} = \mathbf{f}(\mathbf{x}_Q + \tilde{\mathbf{x}}) \tag{5.25}$$

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} \tag{5.26}$$

where

$$\mathbf{A} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_Q} \tag{5.27}$$

is called the *jacobian matrix* of $\mathbf{f}(\mathbf{x})$ evaluated at the equilibrium state $\mathbf{x}_Q \triangleq (x_{1Q}, x_{2Q})$.

It is reasonable to expect that the trajectories in a small neighborhood of Q of Eq. (5.23) are "close" to those in a small neighborhood of the origin ($\tilde{x}_1 = 0, \tilde{x}_2 = 0$) of Eq. (5.24). This "closeness" property is made precise in the following theorem whose proof can be found in Hartman.[30]

Phase portrait linearization property

Assumption: $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ have *continuous* first-order partial derivatives in a neighborhood of the equilibrium state (x_{1Q}, x_{2Q}).

Conclusions: If the origin of the linearized state equation is a *stable* (respectively, *unstable*) *node*, a *stable* (respectively, *unstable*) *focus*, or a *saddle point*, then the trajectories in a small neighborhood of (x_{1Q}, x_{2Q}) of the associated *nonlinear* state equation will also behave "like" a *stable* (respectively, *unstable*) *node*, *stable* (respectively, *unstable*) *focus*, or a *saddle point*.[31]

Moreover, in the case of a *stable* (respectively, *unstable*) *node* or a *saddle point*, the slow and fast eigenvectors of $\mathbf{A}$ in Eq. (5.27) determine the *limiting slope* of the trajectories near $\mathbf{x}_Q$ for the *nonlinear* state equation in the same way depicted in Figs. 3.3 to 3.5.

[30] P. Hartman, *Ordinary Differential Equation*, Wiley, New York, 1964. (The generalization of the *phase portrait linearization property* to higher dimensions is usually referred to as *Hartman's theorem*.)

[31] Roughly speaking, this property asserts that if the trajectories of the *nonlinear* state equation [Eq. (4.2)] were drawn on a rubber sheet, then those trajectories restricted to a sufficiently small neighborhood of $\mathbf{x}_Q$ could be made to *coincide* with those of the *linearized* state equation [Eq. (5.24)] by stretching the rubber sheet in an appropriate way.

The Effect of Small Nonlinear Terms

Is it really safe to neglect the quadratic terms in (1)? In other words, does the linearized system give a qualitatively correct picture of the phase portrait near (x^*, y^*)? The answer is *yes, as long as the fixed point for the linearized system is not one of the borderline cases* discussed in Section 5.2. In other words, if the linearized system predicts a saddle, node, or a spiral, then the fixed point *really is* a saddle, node, or spiral for the original nonlinear system. See Andronov et al. (1973) for a proof of this result, and Example 6.3.1 for a concrete illustration.

The borderline cases (centers, degenerate nodes, stars, or non-isolated fixed points) are much more delicate. They can be altered by small nonlinear terms, as we'll see in Example 6.3.2 and in Exercise 6.3.11.

EXAMPLE 6.3.2:

Consider the system

$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

where a is a parameter. Show that the linearized system *incorrectly* predicts that the origin is a center for all values of a, whereas in fact the origin is a stable spiral if $a < 0$ and an unstable spiral if $a > 0$.

Solution: To obtain the linearization about $(x^*, y^*) = (0,0)$, we can either compute the Jacobian matrix directly from the definition, or we can take the following shortcut. For any system with a fixed point at the origin, x and y represent deviations from the fixed point, since $u = x - x^* = x$ and $v = y - y^* = y$; hence we can linearize by simply omitting nonlinear terms in x and y. Thus the linearized system is $\dot{x} = -y$, $\dot{y} = x$. The Jacobian is

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which has $\tau = 0$, $\Delta = 1 > 0$, so the origin is always a center, according to the linearization.

To analyze the nonlinear system, we change variables to ***polar coordinates***. Let $x = r\cos\theta$, $y = r\sin\theta$. To derive a differential equation for r, we note $x^2 + y^2 = r^2$, so $x\dot{x} + y\dot{y} = r\dot{r}$. Substituting for $\dot{x}$ and $\dot{y}$ yields

$$\begin{aligned} r\dot{r} &= x\left(-y + ax(x^2 + y^2)\right) + y\left(x + ay(x^2 + y^2)\right) \\ &= a(x^2 + y^2)^2 \\ &= ar^4. \end{aligned}$$

Hence $\dot{r} = ar^3$. In Exercise 6.3.12, you are asked to derive the following differential equation for θ:

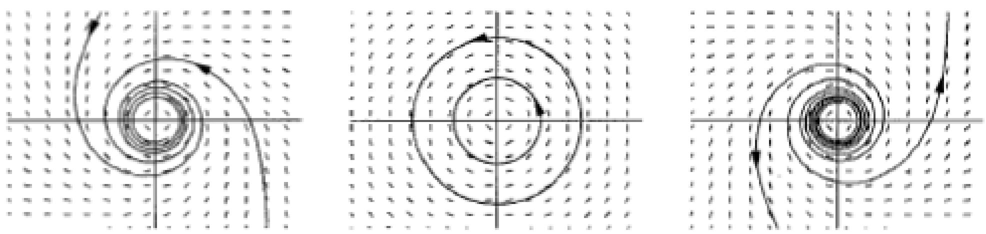
$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}.$$

After substituting for $\dot{x}$ and $\dot{y}$ we find $\dot{\theta} = 1$. Thus in polar coordinates the original system becomes

$$\dot{r} = ar^3$$
$$\dot{\theta} = 1.$$

The system is easy to analyze in this form, because the radial and angular motions are independent. All trajectories rotate about the origin with constant angular velocity $\dot{\theta} = 1$.

The radial motion depends on a, as shown in Figure 6.3.2.



$a < 0$ $\qquad$ $a = 0$ $\qquad$ $a > 0$

Figure 6.3.2

If $a < 0$, then $r(t) \to 0$ monotonically as $t \to \infty$. In this case, the origin is a stable spiral. (However, note that the decay is extremely slow, as suggested by the computer-generated trajectories shown in Figure 6.3.2.) If $a = 0$, then $r(t) = r_0$ for all t and the origin is a center. Finally, if $a > 0$, then $r(t) \to \infty$ monotonically and the origin is an unstable spiral.

We can see now why centers are so delicate: all trajectories are required to close *perfectly* after one cycle. The slightest miss converts the center into a spiral. ■

Similarly, stars and degenerate nodes can be altered by small nonlinearities, but unlike centers, *their stability doesn't change.* For example, a stable star may be changed into a stable spiral (Exercise 6.3.11) but not into an unstable spiral. This is plausible, given the classification of linear systems in Figure 5.2.8: stars and degenerate nodes live squarely in the stable or unstable region, whereas centers live on the razor's edge between stability and instability.

If we're only interested in *stability*, and not in the detailed geometry of the trajectories, then we can classify fixed points more coarsely as follows:

Robust cases:

Repellers (also called *sources*): both eigenvalues have positive real part.

Attractors (also called *sinks*): both eigenvalues have negative real part.

Saddles: one eigenvalue is positive and one is negative.

Marginal cases:

Centers: both eigenvalues are pure imaginary.

Higher-order and non-isolated fixed points: at least one eigenvalue is zero.

Thus, from the point of view of stability, the marginal cases are those where at least one eigenvalue satisfies $\mathrm{Re}(\lambda) = 0$.

Some definitions

- ***Hyperbolic fixed points***

If $\text{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is often called ***hyperbolic***. (This is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard.) Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. Nonhyperbolic fixed points are the fragile ones.

We've already seen a simple instance of hyperbolicity in the context of vector fields on the line. In Section 2.4 we saw that the stability of a fixed point was accurately predicted by the linearization, *as long as* $f'(x^*) \neq 0$. This condition is the exact analog of $\text{Re}(\lambda) \neq 0$.

These ideas also generalize neatly to higher-order systems. A fixed point of an nth-order system is *hyperbolic* if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $\text{Re}(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. The important *Hartman–Grobman theorem* states that the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here ***topologically equivalent*** means that there is a ***homeomorphism*** (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

- ***Structural stability***

Hyperbolic fixed points also illustrate the important general notion of structural stability. A phase portrait is ***structurally stable*** if its topology cannot be changed by an arbitrarily small perturbation to the vector field. For instance, the phase portrait of a saddle point is structurally stable, but that of a center is not: an arbitrarily small amount of damping converts the center to a spiral.

- ***Nullclines***

Nullclines are the curves where either $dx_1/dt = 0$ or $dx_2/dt = 0$. They indicate where the flow is purely horizontal or vertical. Their intersections are the fixed points of the system.

- ***Basin of attraction***

Given an attracting fixed point $\mathbf{x}^*$, we define its basin of attraction to be the set of initial conditions $\mathbf{x_0}$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$.

- ***Stable (unstable) manifold of a saddle***

Given a saddle point $\mathbf{x}^*$, its stable manifold is defined as the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. Likewise, the unstable manifold of $\mathbf{x}^*$ is the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to -\infty$.

Example: Lotka-Volterra model of competition

In the next few sections we'll consider some simple examples of phase plane analysis. We begin with the classic ***Lotka–Volterra model of competition*** between two species, here imagined to be rabbits and sheep. Suppose that both species are competing for the same food supply (grass) and the amount available is limited. A model (known as Lotka-Volterra model) is the following

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

where

$x(t) =$ population of rabbits,
$y(t) =$ population of sheep

To find the fixed points for the system, we solve $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously. Four fixed points are obtained: $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. To classify them, we compute the Jacobian:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Now consider the four fixed points in turn:

$(0,0)$: Then $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

The eigenvalues are $\lambda = 3, 2$ so $(0,0)$ is an *unstable node*. Trajectories leave the origin parallel to the eigenvector for $\lambda = 2$, i.e. tangential to $\mathbf{v} = (0,1)$, which spans the y-axis. (Recall the general rule: at a node, trajectories are tangential to the slow eigendirection, which is the eigendirection with the smallest $|\lambda|$.) Thus, the phase portrait near $(0,0)$ looks like Figure 6.4.1.

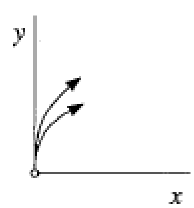


Figure 6.4.1

$(0,2)$: Then $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$.

This matrix has eigenvalues $\lambda = -1, -2$, as can be seen from inspection, since the matrix is triangular. Hence the fixed point is a *stable node*. Trajectories approach along the eigendirection associated with $\lambda = -1$; you can check that this direction is spanned by $\mathbf{v} = (1, -2)$. Figure 6.4.2 shows the phase portrait near the fixed point $(0, 2)$.

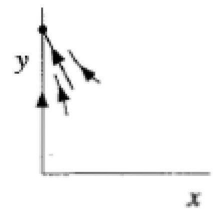


Figure 6.4.2

$(3,0)$: Then $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ and $\lambda = -3, -1$.

This is also a *stable node*. The trajectories approach along the slow eigendirection spanned by $\mathbf{v} = (3, -1)$, as shown in Figure 6.4.3.

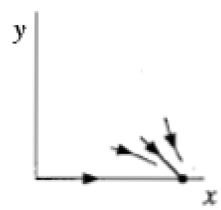


Figure 6.4.3

$(1,1)$: Then $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, which has $\tau = -2$, $\Delta = -1$, and $\lambda = -1 \pm \sqrt{2}$.

Hence this is a *saddle point*. As you can check, the phase portrait near $(1,1)$ is as shown in Figure 6.4.4.

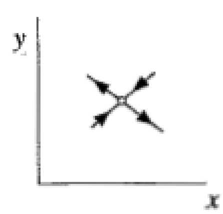


Figure 6.4.4

Combining Figures 6.4.1–6.4.4, we get Figure 6.4.5, which already conveys a good sense of the entire phase portrait. Furthermore, notice that the x and y axes contain straight-line trajectories, since $\dot{x}=0$ when $x=0$, and $\dot{y}=0$ when $y=0$.

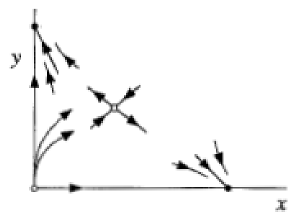


Figure 6.4.5

Now we use common sense to fill in the rest of the phase portrait (Figure 6.4.6). For example, some of the trajectories starting near the origin must go to the stable node on the x-axis, while others must go to the stable node on the y-axis. In between, there must be a special trajectory that can't decide which way to turn, and so it dives into the saddle point. This trajectory is part of the ***stable manifold*** of the saddle, drawn with a heavy line in Figure 6.4.6.

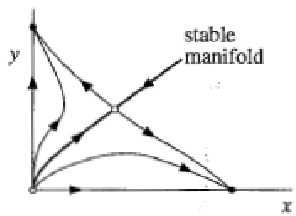


Figure 6.4.6

The other branch of the stable manifold consists of a trajectory coming in "from infinity." A computer-generated phase portrait (Figure 6.4.7) confirms our sketch.

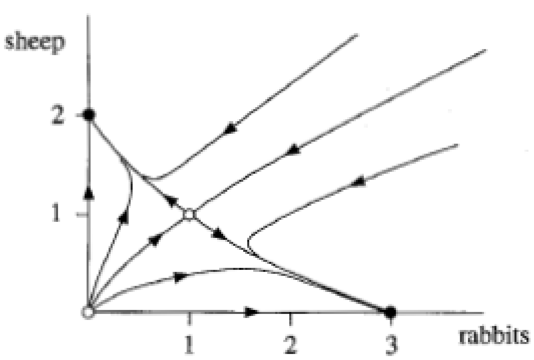


Figure 6.4.7

The phase portrait has an interesting biological interpretation. It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy occurs in other models of competition and has led biologists to formulate the *principle of competitive exclusion*, which states that two species competing for the same limited resource typically cannot coexist. See Pianka (1981) for a biological discussion, and Pielou (1969), Edelstein–Keshet (1988), or Murray (1989) for additional references and analysis.

Our example also illustrates some general mathematical concepts. Given an attracting fixed point $\mathbf{x}^*$, we define its ***basin of attraction*** to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. For instance, the basin of attraction for the node at $(3,0)$ consists of all the points lying below the stable manifold of the saddle. This basin is shown as the shaded region in Figure 6.4.8.

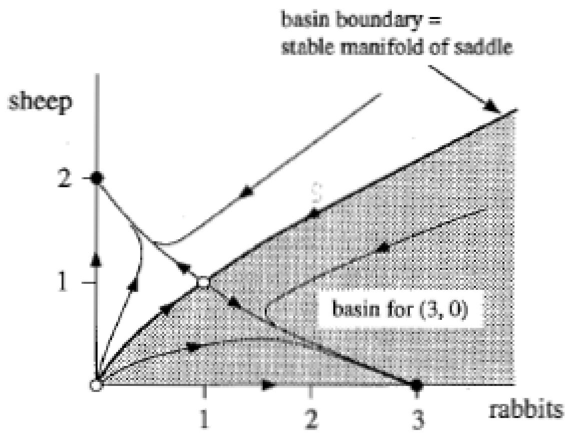


Figure 6.4.8

Because the stable manifold separates the basins for the two nodes, it is called the ***basin boundary***. For the same reason, the two trajectories that comprise the stable manifold are traditionally called ***separatrices***. Basins and their boundaries are important because they partition the phase space into regions of different long-term behavior.