Qualitative behaviour of linear systems

(from Strogatz, Nonlinear dynamics and chaos, Perseus Books, Cambridge, Ma, 1994, Ch. 5)

Let us consider a simple second-order autonomous linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{1}$$

Let us denote with λ_1 and λ_2 its eigenvalues and with $\mathbf{v}_1$ and $\mathbf{v}_2$ the corresponding eigenvectors. For the moment, let us suppose the eigenvalues to be real.

Real eigenvalues

The typical situation is for the eigenvalues to be distinct: $\lambda_1 \neq \lambda_2$. In this case, a theorem of linear algebra states that the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and hence span the entire plane (Figure 5.2.1). In particular, any initial condition $\mathbf{x}_0$ can be written as a linear combination of eigenvectors, say $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

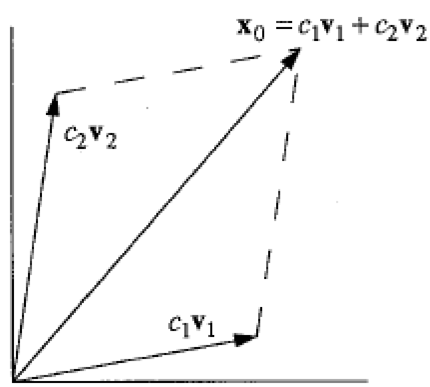


Figure 5.2.1

This observation allows us to write down the general solution for $\mathbf{x}(t)$—it is simply

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \,. \tag{6}$$

Why is this the general solution? First of all, it is a linear combination of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$, and hence is itself a solution. Second, it satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and so by the existence and uniqueness theorem, it is the *only* solution. (See Section 6.2 for a general statement of the existence and uniqueness theorem.)

Phase portrait for the case of **negative distinct eigenvalues** ($\lambda_2 < \lambda_1 < 0$): both solutions decay exponentially. Note that if the initial value $\mathbf{x}(0)$ is on an eigenvector. then the trajectory is a straight line coincident with the eigenvector. The equilibrium point is a **stable node**.

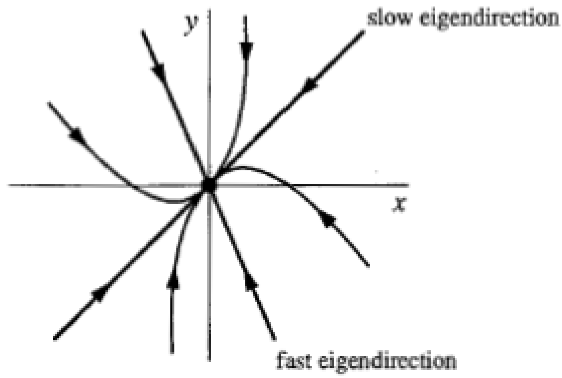
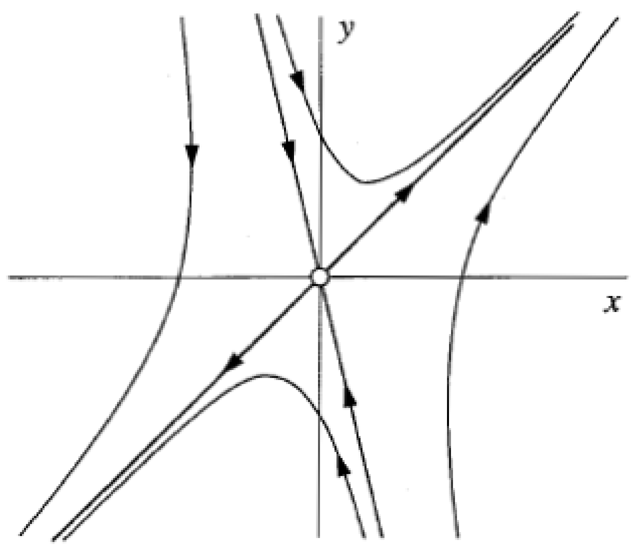


Figure 5.2.3 **Stable node**

Trajectories typically approach the origin tangent to the **slow** eigendirection**,** defined as the direction spanned by the eigenvector with the smaller $|\lambda|$. If the eigenvalues are **distinct positive**, then both solutions grow exponentially: we obtain a phase portrait similar to that of Fig. 5.2.3, with all the arrows reversed (the origin is an **unstable node**).

Phase portrait for the case of two **real eigenvalues of opposite sign** ($\lambda_1 > 0$, $\lambda_2 < 0$): the first eigensolution grows exponentially, and the second eigensolution decays. This means that the origin is a **saddle point** and the stable and the unstable manifold are the lines corresponding to the decaying and growing eigensolution, respectively.



Saddle point

Complex eigenvalues

In the case of a second order linear system with complex eigenvalues $\alpha \pm j\omega_d$ the solution can be written in the form:

$$\begin{aligned} x_1(t) &= A_1 e^{\alpha t} \cos\left(\omega_d t + \varphi_1\right) \\ x_2(t) &= A_2 e^{\alpha t} \cos\left(\omega_d t + \varphi_2\right) \end{aligned} \tag{2}$$

with the constant A_1, φ_1, A_2, and φ_2 determined by the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$. There are two cases to be considered.

1) $\alpha = 0$: imaginary eigenvalues

If $\alpha = 0$, then all the solutions are periodic, with period $T=2\pi/\omega_d$. The trajectories are closed orbits around the fixed point $\mathbf{x} = 0$, with their amplitude determined by the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$. So we have a continuous of closed orbits. The origin is called a **center** (Fig. 5.2.4a)

2) $\alpha \neq 0$: complex eigenvalues

If $\alpha \neq 0$, then x_1 and x_2 are exponentially decaying oscillations if $\alpha < 0$ and growing oscillations if $\alpha > 0$. The trajectories are logarithmic spirals around the origin.

If $\alpha < 0$ they shrink to the origin as $t \to \infty$ as shown in Fig. 5.2.4b. The equilibrium state $\mathbf{x} = 0$ in this case is called a **stable focus** (or **stable spiral**).

If $\alpha > 0$ the trajectories are logarithmic spirals which expand toward infinity as $t \to \infty$. The equilibrium state $\mathbf{x} = 0$ in this case is called an **unstable focus** (or **unstable spiral**).

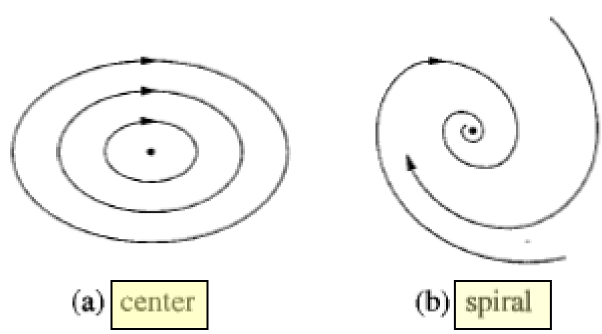


Figure 5.2.4

EXAMPLE 5.2.5:

In our analysis of the general case, we have been assuming that the eigenvalues are distinct. What happens if the eigenvalues are *equal*?

Solution: Suppose $\lambda_1 = \lambda_2 = \lambda$. There are two possibilities: either there are two independent eigenvectors corresponding to λ, or there's only one.

If there are two independent eigenvectors, then they span the plane and so *every vector is an eigenvector with this same eigenvalue* λ. To see this, write an arbitrary vector $\mathbf{x}_0$ as a linear combination of the two eigenvectors: $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then

$$A\mathbf{x}_0 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda\mathbf{v}_1 + c_2\lambda\mathbf{v}_2 = \lambda\mathbf{x}_0$$

so $\mathbf{x}_0$ is also an eigenvector with eigenvalue λ. Since multiplication by A simply stretches every vector by a factor λ, the matrix must be a multiple of the identity:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Then if $\lambda \neq 0$, all trajectories are straight lines through the origin ($\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$) and the fixed point is a *star node* (Figure 5.2.5).

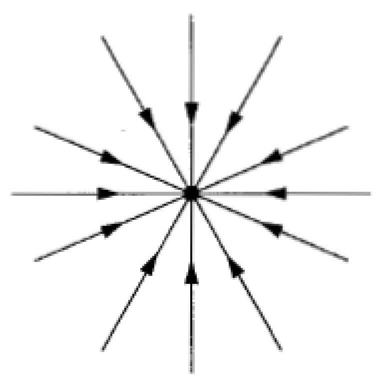


Figure 5.2.5