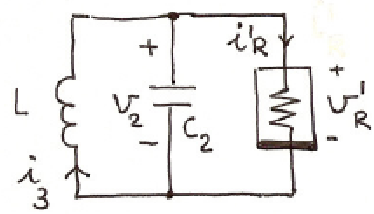
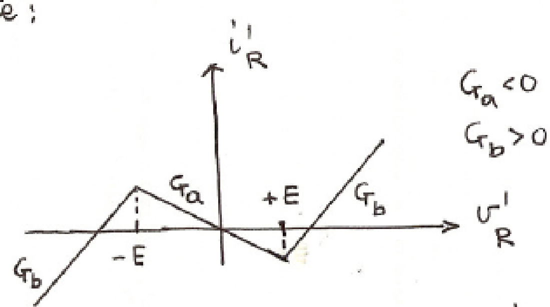
Nonlinear state equation formulation

Example # 1

Si scrivano le equazioni di stato per il circuito seguente:

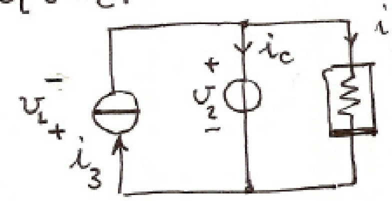




$$i'_R = G_b v'_R + \frac{1}{2}\left(\left|v'_R + E\right| - \left|v'_R - E\right|\right)\left(G_a - G_b\right)$$

Assumo v_2 e i_3 come variabili di stato

Sostituisco il condensatore con un generatore di tensione e l'induttore con un generatore di corrente e calcolo le grandezze "coniugate" v_L e i_C:



$i'_R = g(v'_R) = g(v_2)$

$$v_L = -v_2 \quad \longrightarrow \quad L\frac{di_3}{dt} = -v_2 \quad ; \quad \frac{di_3}{dt} = -\frac{1}{L}v_2$$

$$i_C = i_3 - i'_R \quad \rightarrow \quad C_2\frac{dv_2}{dt} = +i_3 - g(v_2); \quad \frac{dv_2}{dt} = -\frac{1}{C_2}i_3 - \frac{1}{C_2}g(v_2)$$

In definitiva le eq. di stato sono:

$$\begin{cases} \dfrac{dv_2}{dt} = -\dfrac{1}{C_2}g(v_2) + \dfrac{1}{C_2}i_3 \\ \dfrac{di_3}{dt} = -\dfrac{1}{L}v_2 \end{cases}$$

Example # 2 (Hasler-Neirynck, p. 113)

Let us consider the Wien-bridge oscillator of figure 2.56 (vol. VIII, subsec. 5.3.5). The associated resistive circuit is shown in figure 2.57. According to (2.146), the currents i_1 and i_2 must be expressed as a function of the voltages v_{C1} and v_{C2}. If the two-port were linear, the admittance matrix should be calculated.

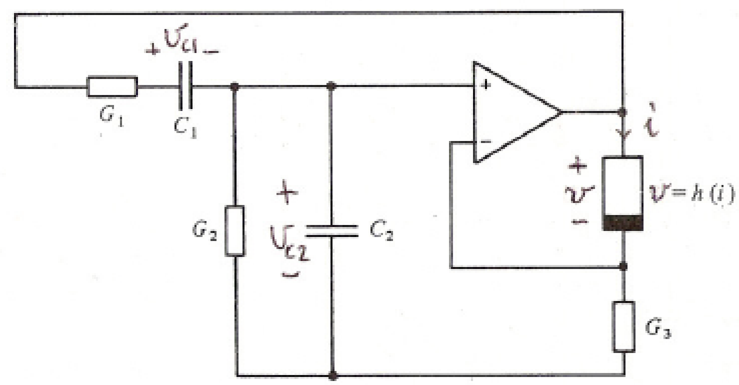


Fig. 2.56

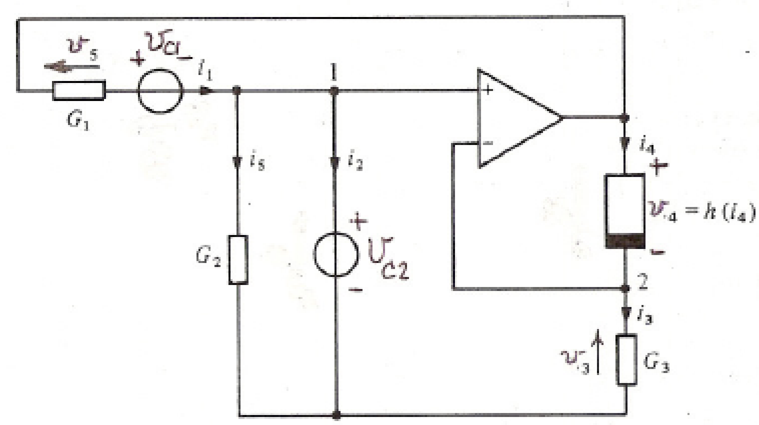


Fig. 2.57

Si noti che $v_3 = v_{c2}$ e che $i_4 = i_3$

Quindi $v_4 = h(i_4) = h(v_{c2} G_3)$

Ora posso calcolare i_1:

$$i_1 = \left[(v_3 + v_4) - (v_{c1} + v_{c2})\right] G_1 =$$

$$= \left[\cancel{v_{c2}} + h(v_{c2} G_3) - v_{c1} - \cancel{v_{c2}}\right] G_1 =$$

$$= -v_{c1} G_1 + G_1 h(v_{c2} G_3)$$

Ma $i_1 = C_1 \dfrac{dv_{c1}}{dt}$ e quindi:

$$\frac{dv_{c1}}{dt} = -\frac{1}{R_1 C_1} v_{c1} + \frac{1}{R_1 C_1} h(v_{c2} G_3)$$

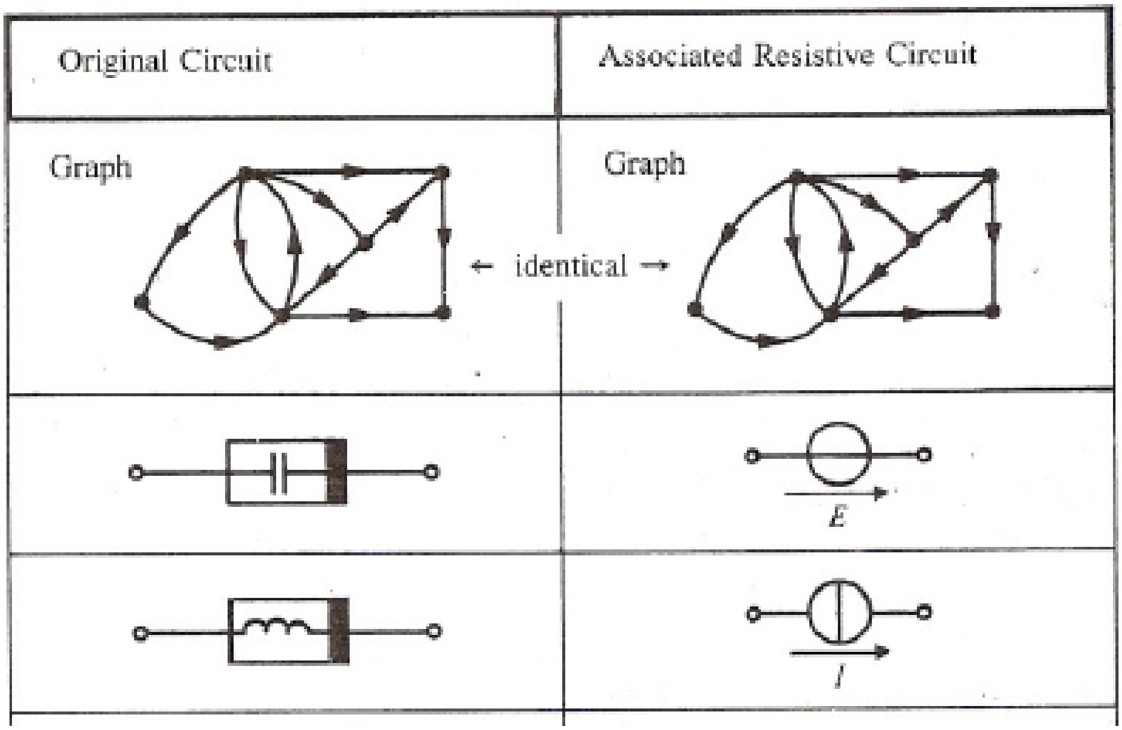
Infine $i_2 = i_1 - i_5 = i_1 - v_{c2} G_2 =$

$$= -v_{c1} G_1 - v_{c2} G_2 + G_1 h(v_{c2} G_3)$$

Le eq. di stato sono:

$$\begin{cases} \dfrac{dv_{c1}}{dt} = -\dfrac{1}{R_1 C_1} v_{c1} + \dfrac{1}{R_1 C_1} h(v_{c2} G_3) \\[2ex] \dfrac{dv_{c2}}{dt} = -\dfrac{1}{R_1 C_2} v_{c1} - \dfrac{1}{R_2 C_2} v_{c2} + \dfrac{1}{R_1 C_2} h(v_{c2} G_3) \end{cases}$$

Nonlinear state equation formulation: step 1 (Hasler-Neirynck, p. 112)



Nonlinear state equation formulation: step 2 (CDK, p. 405)

Capacitor

1. If the capacitor is *voltage-controlled*, i.e.,

$$q_{Cj} = \hat{q}_{Cj}(v_{Cj}) \qquad (4.14a)$$

   then choose *capacitor voltage* v_{Cj} as the state variable.

2. If the capacitor is *charge-controlled*, i.e.,

$$v_{Cj} = \hat{v}_{Cj}(q_{Cj}) \qquad (4.14b)$$

   then choose *capacitor charge* q_{Cj} as the state variable.

Inductor

1. If the inductor is *current-controlled*, i.e.,

$$\phi_{Lj} = \hat{\phi}_{Lj}(i_{Lj}) \qquad (4.15a)$$

   then choose *inductor current* i_{Lj} as the state variable.

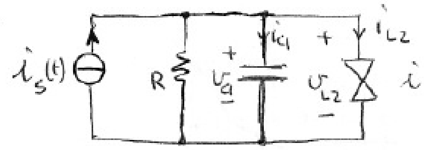
2. If the inductor is *flux-controlled*, i.e.,

$$i_{Lj} = \hat{i}_{Lj}(\phi_{Lj}) \qquad (4.15b)$$

   then choose *inductor flux* ϕ_{Lj} as the state variable.

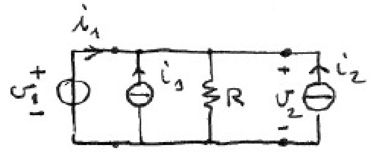
Example # 3

Si consideri questo nuovo esempio: (CDK, p. 401)



$i_{L2} = I_0 \,\mathrm{sen}\, k\varphi_{L2}$
(Josephson junction)

Assumo v_{C1} e φ_{L2} come variabili di stato.
Creo il circuito resistivo associato:



con $i_1 = -i_{C1}$ e
$i_2 = -i_{L2}$

Devo calcolare i_1 e v_2 in funzione di v_1 e i_2

$$\begin{cases} i_1 = -i_s + G v_1 - i_2 \\ v_2 = v_1 \end{cases}$$

ma $i_1 = C_1 \, dv_1/dt$ e
$v_2 = d\varphi_2/dt$.

Quindi ho:

$$\begin{cases} \dfrac{dv_{C1}}{dt} = -\dfrac{v_{C1}}{RC_1} - \dfrac{1}{C_1} I_0 \,\mathrm{sen}\, k\varphi_{L2} + \dfrac{1}{C_1} i_s(t) \\ d\varphi_{L2}/dt = v_{C1} \end{cases}$$

General remarks (CDK, p. 364 and 400)

1.

There are two important reasons for writing state equations. *First*, all properly modeled circuits have a well-defined state equation. *Second*, most analytic and numerical methods for solving *nonlinear* differential equations are formulated in terms of the above standard form.

Note that to qualify as a state equation, $\dot{x}_1$ and $\dot{x}_2$ must appear on the left-hand side, and only x_1 and x_2 can appear on the right-hand side, in addition to possibly the independent time variable t.

In the special case where the circuit contains only time-invariant elements and dc independent sources, the time variable does not appear explicitly. In this case, Eq. (4.1) reduces to

Autonomous state equation

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{4.2}$$

Equation (4.2) is usually called an *autonomous* state equation in the literature.[19] Consequently, dynamic circuits containing only linear time-invariant elements and dc sources are called *autonomous circuits*.

2.

6.1.1.3 Non-autonomous dynamical systems

A non-autonomous n-dimensional continuous-time dynamical system may be transformed to an $(n+1)$-dimensional *autonomous* system by appending time as an additional state variable and writing

$$\begin{aligned} \dot{X}(t) &= F(X(t), X_{n+1}(t)) \\ \dot{X}_{n+1}(t) &= 1 \end{aligned} \tag{5}$$