Limit cycles(*)

A ***limit cycle*** is an isolated closed trajectory. *Isolated* means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle (Figure 7.0.1).

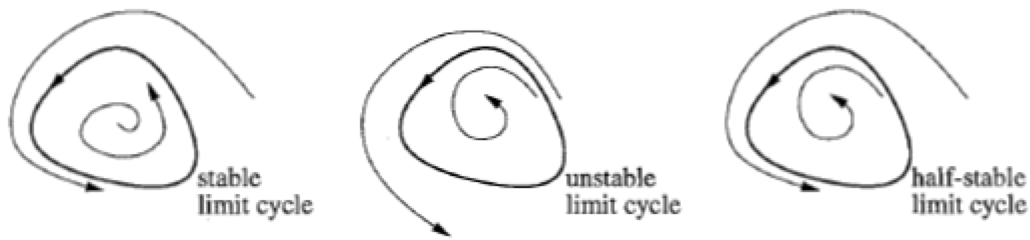

Figure 7.0.1

If all neighboring trajectories approach the limit cycle, we say the limit cycle is *stable* or *attracting*. Otherwise the limit cycle is *unstable*, or in exceptional cases, *half-stable*.

Stable limit cycles are very important scientifically—they model systems that exhibit self-sustained oscillations. In other words, these systems oscillate even in the absence of external periodic forcing. Of the countless examples that could be given, we mention only a few: the beating of a heart; the periodic firing of a pacemaker neuron; daily rhythms in human body temperature and hormone secretion; chemical reactions that oscillate spontaneously; and dangerous self-excited vibrations in bridges and airplane wings. In each case, there is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it always returns to the standard cycle.

Limit cycles are inherently nonlinear phenomena; they can't occur in linear systems. Of course, a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ can have closed orbits, but they won't be *isolated*; if $\mathbf{x}(t)$ is a periodic solution, then so is $c\mathbf{x}(t)$ for any constant $c \neq 0$. Hence $\mathbf{x}(t)$ is surrounded by a one-parameter family of closed orbits (Figure 7.0.2).

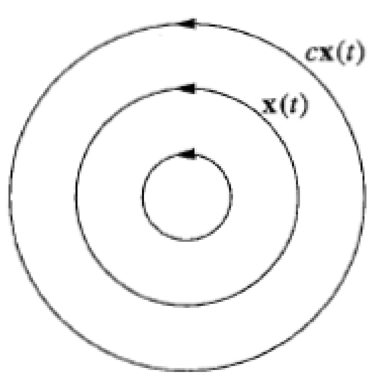

Figure 7.0.2

Consequently, the amplitude of a linear oscillation is set entirely by its initial conditions; any slight disturbance to the amplitude will persist forever. In contrast, limit cycle oscillations are determined by the structure of the system itself.

(*) S. H. Strogatz, Nonlinear Dynamics and Chaos, Ch. 7, Perseus Books, Cambridge, MA, 1994

Van der Pol oscillator

Source: Philips Company Archives

Balthasar Van der Pol was born on 27 January 1889, at Utrecht, Holland. He studied experimental physics with J. A. Fleming and Sir J. J. Thompson in England, and was to H. A. Lorentz in the Conservator Physical Laboratory, Holland. He received his Doctor of Physics degree from Utrecht in 1920.

Van der Pol initiated modern experimental dynamics in the laboratory during the 1920's and 1930's. He investigated electrical circuits employing vacuum tubes and found that they have stable oscillations, now called limit cycles. When these circuits are driven with a signal whose frequency is near that of the limit cycle, the resulting periodic response shifts its frequency to that of the driving signal. That is to say, the circuit becomes "entrained" to the driving signal. The waveform, or signal shape, however, can be quite complicated and contain a rich structure of harmonics and subharmonics.

In the September 1927 issue of the British journal *Nature*, Van der Pol and his colleague van der Mark reported that an "irregular noise" was heard at certain driving frequencies between the natural entrainment frequencies. By reconstructing his electronic tube circuit, we now know that they had discovered deterministic chaos. Their paper is probably one of the first experimental reports of chaos -something that they failed to pursue in more detail.

Van der Pol built a number of electronic circuit models of the human heart to study the range of stability of heart dynamics. His investigations with adding an external driving signal were analogous to the situation in which a real heart is driven by a pacemaker. He was interested in finding out, using his entrainment work, how to stabilize a heart's irregular beating or "arrhythmias".

Van der Pol was one of the founders, and for many years president, of "Het Nederlandsch Radiogenootschap," and member of the Union Internationale de Radio Diffusion, and he Union Radio Scientifique Internationale. He joined the Institute of Radio Engineers in 1920, and became a Fellow in 1929. He received the IRE Medal of Honor in 1935 "For his fundamental studies and contributions in the field of circuit theory and electromagnetic wave propagation phenomena." Van der Pol passed away in 1959.

A less transparent example, but one that played a central role in the development of nonlinear dynamics, is given by the ***van der Pol equation***

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \qquad (2)$$

where $\mu \geq 0$ is a parameter. Historically, this equation arose in connection with the nonlinear electrical circuits used in the first radios (see Exercise 7.1.6 for the circuit). Equation (2) looks like a simple harmonic oscillator, but with a ***nonlinear damping*** term $\mu(x^2 - 1)\dot{x}$. This term acts like ordinary positive damping for $|x| > 1$, but like *negative* damping for $|x| < 1$. In other words, it causes large-amplitude oscillations to decay, but it pumps them back up if they become too small.

As you might guess, the system eventually settles into a self-sustained oscillation where the energy dissipated over one cycle balances the energy pumped in. This idea can be made rigorous, and with quite a bit of work, one can prove that *the van der Pol equation has a unique, stable limit cycle for each* $\mu > 0$. This result follows from a more general theorem discussed in Section 7.4.

To give a concrete illustration, suppose we numerically integrate (2) for $\mu = 1.5$, starting from $(x, \dot{x}) =$ (0.5, 0) at $t = 0$. Figure 7.1.4 plots the solution in the phase plane and Figure 7.1.5 shows the graph of $x(t)$. Now, in contrast to Example 7.1.1, the limit cycle is not a circle and the stable waveform is not a sine wave. ■

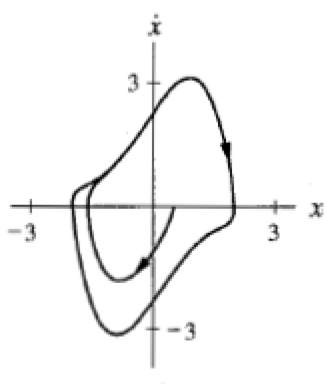

Figure 7.1.4

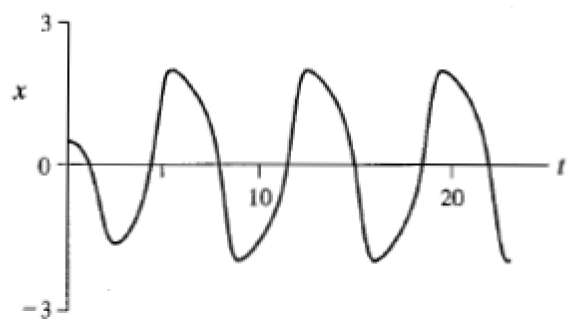

Figure 7.1.5

Note: Historically, Eq. (2) is known as Van der Pol equation. It is equivalent to the following coupled first order differential equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \mu(1 - x_1^2)x_2 \end{cases}$$

For the meaning of the state variables, see later in this report.

State equations of the Van der Pol oscillator

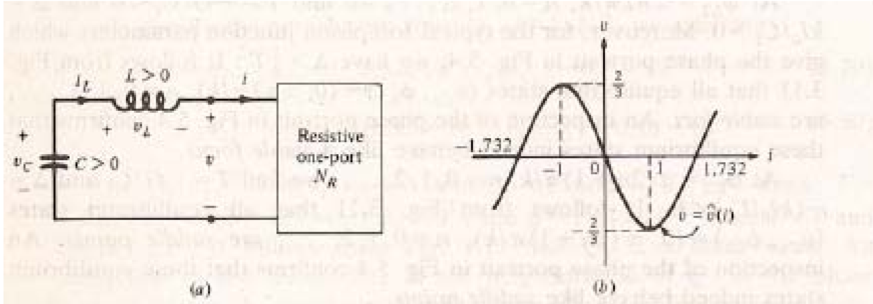

Figure 1 Van der Pol oscillator: (a) circuit; (b) characteristic of resistor N_R

Assuming that

$$v = \frac{1}{3} i_L^3 - i_L$$

the state equations of the Van der Pol oscillator shown in Fig. 1 become:

$$\begin{cases} \dfrac{dv_C}{dt} = -\dfrac{1}{C} i_L \\ \dfrac{di_L}{dt} = \dfrac{1}{L} v_C - \dfrac{1}{L}\left[\dfrac{1}{3} i_L^3 - i_L\right] \end{cases}$$

To keep the independent parameters to a minimum, let us introduce the following *scaled* time variable:

$$\tau \triangleq \frac{1}{\sqrt{LC}} t$$

Observe that

$$dt = \sqrt{LC}\, d\tau$$

Substituting into the state equations above, we obtain the following equivalent state equations in terms of the dimensionless time τ

$$\begin{cases} \dfrac{dv_C}{d\tau} = -\dfrac{1}{\mu} i_L \\ \dfrac{di_L}{d\tau} = \mu\left[v_C - \dfrac{1}{3} i_L^3 + i_L\right] \end{cases} \quad (1)$$

where

$$\mu \triangleq \sqrt{\frac{C}{L}} \quad (2)$$

Remark: Relationship between Eq. (1) and the *"classical"* 2nd-order Van del Pol equation

The state equations (1) lead to the classical 2nd-order Van der Pol differential equation. To this end, let us derive the second equation of (1) with respect to τ

$$\frac{d^2 i_L}{d\tau^2} = \mu \frac{dv_C}{d\tau} - \mu\left(i_L^2 - 1\right)\frac{di_L}{d\tau} \tag{3}$$

Using the first equation of (1) it follows that $\frac{dv_C}{d\tau} = -\frac{1}{\mu} i_L$ and hence:

$$\frac{d^2 i_L}{d\tau^2} = -i_L - \mu\left(i_L^2 - 1\right)\frac{di_L}{d\tau} \tag{4}$$

By changing i_L to x and τ to t, we get :

$$\frac{d^2 x}{dt^2} + \mu\left(x^2 - 1\right)\frac{dx}{dt} + x = 0 \tag{5}$$

In the literature, eq. (5) is usually referred to as the ***Van der Pol equation*** and coincides with that reported at the beginning of p. 2 of this report.
Coming back to the electrical circuit, *x(t)* represents the evolution of the inductor current i_L as a function of the normalized time $\tau = t/\sqrt{LC}$.

Poincaré-Bendixson theorem

Now that we know how to rule out closed orbits, we turn to the opposite task: finding methods to *establish that closed orbits exist* in particular systems. The following theorem is one of the few results in this direction. It is also one of the key theoretical results in nonlinear dynamics, because it implies that chaos can't occur in the phase plane, as discussed briefly at the end of this section.

Poincaré–Bendixson Theorem: Suppose that:

(1) R is a closed, bounded subset of the plane;

(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

(3) R does not contain any fixed points; and

(4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time (Figure 7.3.1).

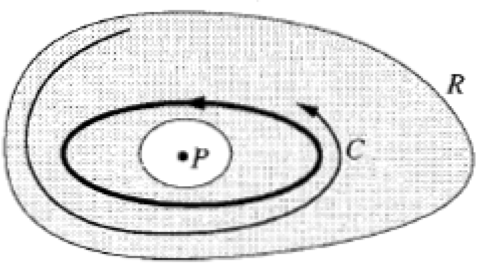

Figure 7.3.1

Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R *contains a closed orbit* (shown as a heavy curve in Figure 7.3.1).

The proof of this theorem is subtle, and requires some advanced ideas from topology. For details, see Perko (1991), Coddington and Levinson (1955), Hurewicz (1958), or Cesari (1963).

In Figure 7.3.1, we have drawn R as a ring-shaped region because any closed orbit must encircle a fixed point (P in Figure 7.3.1) and no fixed points are allowed in R.

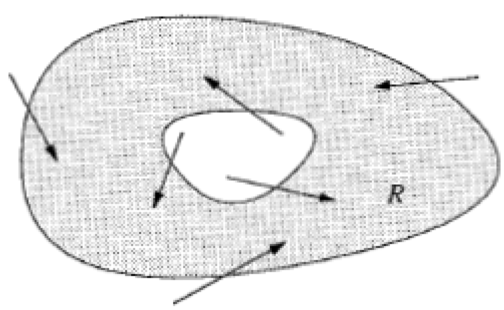

Figure 7.3.2

When applying the Poincaré–Bendixson theorem, it's easy to satisfy conditions (1)–(3); condition (4) is the tough one. How can we be sure that a confined trajectory C exists? The standard trick is to construct a ***trapping region*** R, i.e., a closed connected set such that the vector field points "inward" everywhere on the boundary of R (Figure 7.3.2). Then *all* trajectories in R are confined. If we can also arrange that there are no fixed points in R, then the Poincaré–Bendixson theorem ensures that R contains a closed orbit.

The Poincaré–Bendixson theorem can be difficult to apply in practice. One convenient case occurs when the system has a simple representation in polar coordinates.

No Chaos in the Phase Plane

The Poincaré–Bendixson theorem is one of the central results of nonlinear dynamics. It says that the dynamical possibilities in the phase plane are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. Nothing more complicated is possible.

This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems ($n \geq 3$), the Poincaré–Bendixson theorem no longer applies, and something radically new can happen: trajectories may wander around forever in a bounded region without settling down to a fixed point or a closed orbit. In some cases, the trajectories are attracted to a complex geometric object called a *strange attractor*, a fractal set on which the motion is aperiodic and sensitive to tiny changes in the initial conditions. This sensitivity makes the motion unpredictable in the long run. We are now face to face with *chaos*. We'll discuss this fascinating topic soon enough, but for now you should appreciate that the Poincaré–Bendixson theorem implies that chaos can never occur in the phase plane.

Lienard's theorem

In the early days of nonlinear dynamics, say from about 1920 to 1950, there was a great deal of research on nonlinear oscillations. The work was initially motivated by the development of radio and vacuum tube technology, and later it took on a mathematical life of its own. It was found that many oscillating circuits could be modeled by second-order differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \qquad (1)$$

now known as ***Liénard's equation***. This equation is a generalization of the van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ mentioned in Section 7.1. It can also be interpreted mechanically as the equation of motion for a unit mass subject to a nonlinear damping force $-f(x)\dot{x}$ and a nonlinear restoring force $-g(x)$.

Liénard's equation is equivalent to the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - f(x_1)x_2 \end{cases} \qquad (2)$$

The following theorem states that this system has a unique, stable limit cycle under appropriate hypotheses on f and g. For a proof, see Jordan and Smith (1987), Grimshaw (1990), or Perko (1991).

Liénard's Theorem: Suppose that $f(x)$ and $g(x)$ satisfy the following conditions:

(1) $f(x)$ and $g(x)$ are continuously differentiable for all x;

(2) $g(-x) = -g(x)$ for all x (i.e., $g(x)$ is an *odd* function);

(3) $g(x) > 0$ for $x > 0$;

(4) $f(-x) = f(x)$ for all x (i.e., $f(x)$ is an *even* function);

(5) The odd function $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system (2) has a unique, stable limit cycle surrounding the origin in the phase plane.

This result should seem plausible. The assumptions on $g(x)$ mean that the restoring force acts like an ordinary spring, and tends to reduce any displacement, whereas the assumptions on $f(x)$ imply that the damping is negative at small $|x|$ and positive at large $|x|$. Since small oscillations are pumped up and large oscillations are damped down, it is not surprising that the system tends to settle into a self-sustained oscillation of some intermediate amplitude.

EXAMPLE 7.4.1:

Show that the van der Pol equation has a unique, stable limit cycle.

Solution: The van der Pol equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ has $f(x) = \mu(x^2 - 1)$ and $g(x) = x$, so conditions (1)–(4) of Liénard's theorem are clearly satisfied. To check condition (5), notice that

$$F(x) = \mu\left(\tfrac{1}{3}x^3 - x\right) = \tfrac{1}{3}\mu x(x^2 - 3).$$

Hence condition (5) is satisfied for $a = \sqrt{3}$. Thus the van der Pol equation has a unique, stable limit cycle. ■