STROGATZ, p. 6
and KENNEDY, 6.1.1.3

Introduction to state equations; existence and uniqueness theorem.

All properly modeled systems have a well-defined state equation of the form:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n). \end{aligned} \tag{1}$$

Here the overdots denote differentiation with respect to t. Thus $\dot{x}_i \equiv dx_i/dt$. The variables $x_1, \ldots, x_n$ might represent concentrations of chemicals in a reactor, populations of different species in an ecosystem, or the positions and velocities of the planets in the solar system. The functions $f_1, \ldots, f_n$ are determined by the problem at hand.
The solution of (1) requires the knowledge of the initial conditions $x_1(0), \ldots, x_n(0)$.
If any of the functions f_i is nonlinear, the system is said nonlinear.
Note that to qualify as a state equation, the derivative must appear on the left-hand side, and only the variables $x_1, \ldots, x_n$ can appear on the right-hand side, in addition to possibly the independent time variable t.

In the reported case, system (1) does not include any explicit time dependence: it is said autonomous.
In vector notation, system (1) is written in the following form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

6.1.1.3 *Non-autonomous dynamical systems*

A non-autonomous n-dimensional continuous-time dynamical system may be transformed to an $(n+1)$-dimensional *autonomous* system by appending time as an additional state variable and writing

$$\begin{aligned} \dot{X}(t) &= F(X(t), X_{n+1}(t)) \\ \dot{X}_{n+1}(t) &= 1 \end{aligned} \tag{3}$$

Existence and Uniqueness Theorem: Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Suppose that $\mathbf{f}$ is continuous and that all its partial derivatives $\partial f_i / \partial x_j$, $i, j = 1, \ldots, n$, are continuous for $\mathbf{x}$ in some open connected set $D \subset \mathbf{R}^n$. Then for $\mathbf{x}_0 \in D$, the initial value problem has a solution $\mathbf{x}(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

In other words, existence and uniqueness of solutions are guaranteed if $\mathbf{f}$ is continuously differentiable. The proof of the theorem is similar to that for the case $n = 1$, and can be found in most texts on differential equations. Stronger versions of the theorem are available, but this one suffices for most applications.

From now on, we'll assume that all our vector fields are smooth enough to ensure the existence and uniqueness of solutions, starting from any point in phase space.

The existence and uniqueness theorem has an important corollary: *different trajectories never intersect.* If two trajectories *did* intersect, then there would be two solutions starting from the same point (the crossing point), and this would violate the uniqueness part of the theorem. In more intuitive language, a trajectory can't move in two directions at once.

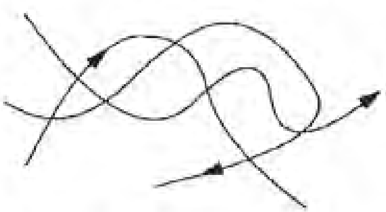

Figure 6.2.1

Because trajectories can't intersect, phase portraits always have a well-groomed look to them. Otherwise they might degenerate into a snarl of criss-crossed curves (Figure 6.2.1). The existence and uniqueness theorem prevents this from happening.

In two-dimensional phase spaces (as opposed to higher-dimensional phase spaces), these results have especially strong topological consequences. For example, suppose there is a closed orbit C in the phase plane. Then any trajectory starting inside C is trapped in there forever (Figure 6.2.2).

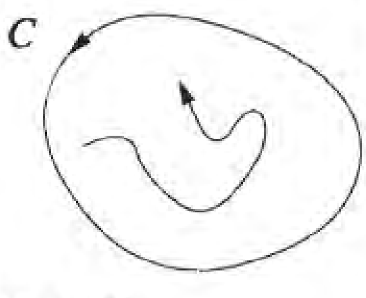

Figure 6.2.2

What is the fate of such a bounded trajectory? If there are fixed points inside C, then of course the trajectory might eventually approach one of them. But what if there *aren't* any fixed points? Your intuition may tell you that the trajectory can't meander around forever—if so, you're right. For vector fields on the plane, the ***Poincaré–Bendixson theorem*** states that if a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must eventually approach a closed orbit.