First-order nonlinear systems

(from Strogatz, Nonlinear dynamics and chaos, Perseus Books, Cambridge, Ma, 1994, Ch. 2)

2.1 A Geometric Way of Thinking

Pictures are often more helpful than formulas for analyzing nonlinear systems. Here we illustrate this point by a simple example. Along the way we will introduce one of the most basic techniques of dynamics: *interpreting a differential equation as a vector field.*

Consider the following nonlinear differential equation:

$$\dot{x} = \sin x. \tag{1}$$

To emphasize our point about formulas versus pictures, we have chosen one of the few nonlinear equations that can be solved in closed form. We separate the variables and then integrate:

$$dt = \frac{dx}{\sin x},$$

which implies

$$t = \int \csc x \, dx$$

$$= -\ln\left|\csc x + \cot x\right| + C.$$

To evaluate the constant C, suppose that $x = x_0$ at $t = 0$. Then $C = \ln\left|\csc x_0 + \cot x_0\right|$. Hence the solution is

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|. \tag{2}$$

This result is exact, but a headache to interpret. For example, can you answer the following questions?

1. Suppose $x_0 = \pi/4$; describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \to \infty$?
2. For an *arbitrary* initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$?

Think about these questions for a while, to see that formula (2) is not transparent.

In contrast, a graphical analysis of (1) is clear and simple, as shown in Figure 2.1.1. We think of t as time, x as the position of an imaginary particle moving along the real line, and $\dot{x}$ as the velocity of that particle. Then the differential equation $\dot{x} = \sin x$ represents a ***vector field*** on the line: it dictates the velocity vector $\dot{x}$ at each x. To sketch the vector field, it is convenient to plot $\dot{x}$ versus x, and then draw arrows on the x-axis to indicate the corresponding velocity vector at each x. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$.

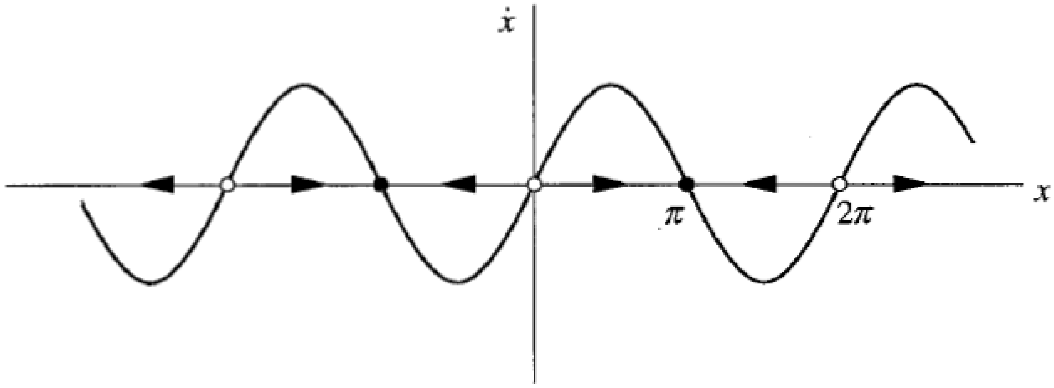

Figure 2.1.1

Here's a more physical way to think about the vector field: imagine that fluid is flowing steadily along the x-axis with a velocity that varies from place to place, according to the rule $\dot{x} = \sin x$. As shown in Figure 2.1.1, the ***flow*** is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At points where $\dot{x} = 0$, there is no flow; such points are therefore called ***fixed points***. You can see that there are two kinds of fixed points in Figure 2.1.1: solid black dots represent ***stable*** fixed points (often called *attractors* or *sinks,* because the flow is toward them) and open circles represent ***unstable*** fixed points (also known as *repellers* or *sources*).

Armed with this picture, we can now easily understand the solutions to the differential equation $\dot{x} = \sin x$. We just start our imaginary particle at x_0 and watch how it is carried along by the flow.

This approach allows us to answer the questions above as follows:

1. Figure 2.1.1 shows that a particle starting at $x_0 = \pi/4$ moves to the right faster and faster until it crosses $x = \pi/2$ (where $\sin x$ reaches its maximum). Then the particle starts slowing down and eventually approaches the stable fixed point $x = \pi$ from the left. Thus, the qualitative form of the solution is as shown in Figure 2.1.2.

   Note that the curve is concave up at first, and then concave down; this corresponds to the initial acceleration for $x < \pi/2$, followed by the deceleration toward $x = \pi$.

2. The same reasoning applies to any initial condition x_0. Figure 2.1.1 shows that if $\dot{x} > 0$ initially, the particle heads to the right and asymptotically approaches the nearest stable fixed point. Similarly, if $\dot{x} < 0$ initially, the particle approaches the nearest stable fixed point to its left. If $\dot{x} = 0$, then x remains constant. The qualitative form of the solution for any initial condition is sketched in Figure 2.1.3.

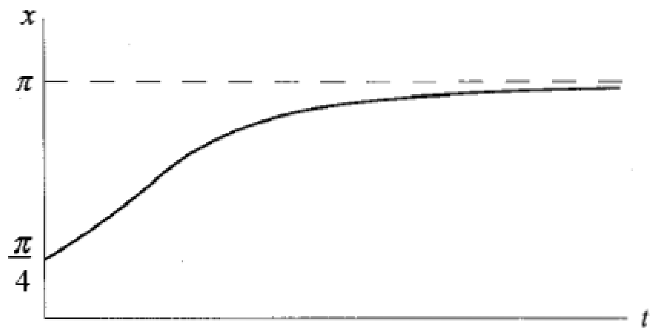

Figure 2.1.2

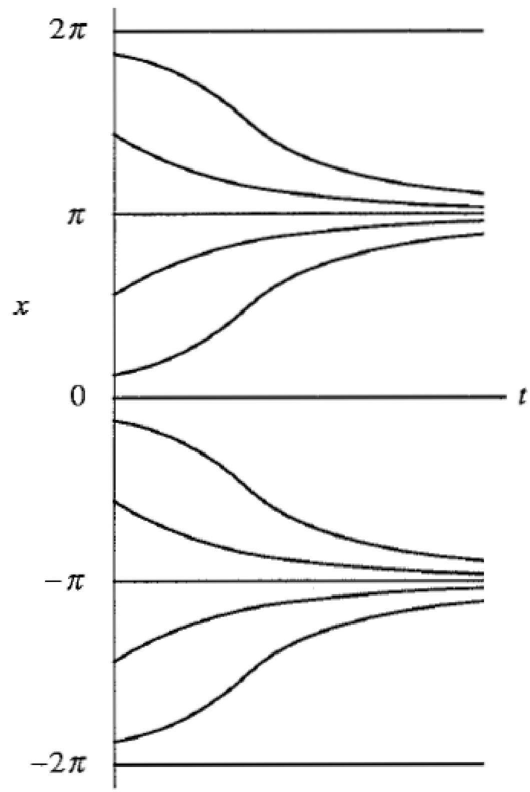

Figure 2.1.3

In all honesty, we should admit that a picture can't tell us certain *quantitative* things: for instance, we don't know the time at which the speed $|\dot{x}|$ is greatest. But in many cases *qualitative* information is what we care about, and then pictures are fine.

2.2 Fixed Points and Stability

The ideas developed in the last section can be extended to any one-dimensional system $\dot{x} = f(x)$. We just need to draw the graph of $f(x)$ and then use it to sketch the vector field on the real line (the x-axis in Figure 2.2.1).

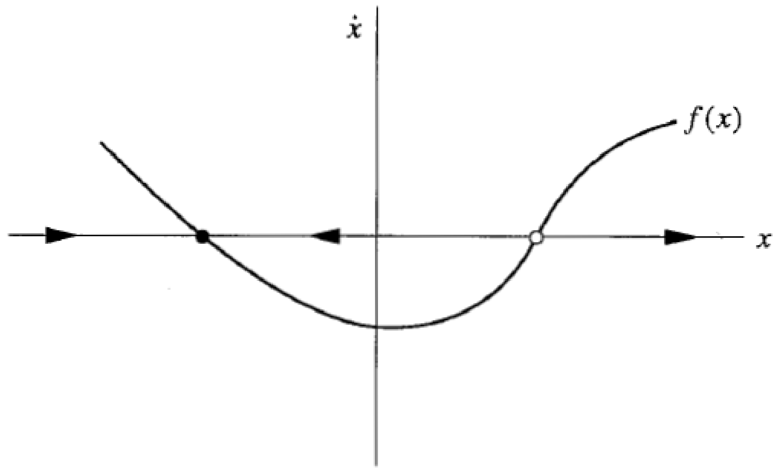

Figure 2.2.1

EXAMPLE 2.2.1:

Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

Solution: Here $f(x) = x^2 - 1$. To find the fixed points, we set $f(x^*) = 0$ and solve for x^*. Thus $x^* = \pm 1$. To determine stability, we plot $x^2 - 1$ and then sketch the vector field (Figure 2.2.2). The flow is to the right where $x^2 - 1 > 0$ and to the left where $x^2 - 1 < 0$. Thus $x^* = -1$ is stable, and $x^* = 1$ is unstable. ■

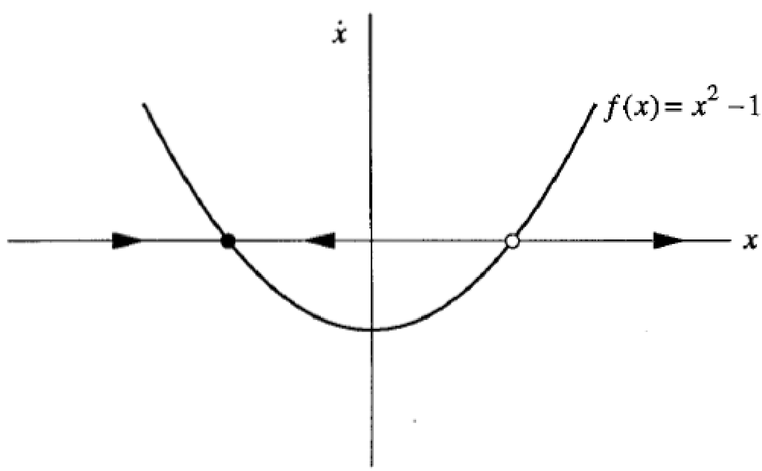

Figure 2.2.2

2.4 Linear Stability Analysis

So far we have relied on graphical methods to determine the stability of fixed points. Frequently one would like to have a more quantitative measure of stability, such as the rate of decay to a stable fixed point. This sort of information may be obtained by *linearizing* about a fixed point, as we now explain.

Let x^* be a fixed point, and let $\eta(t) = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we derive a differential equation for η. Differentiation yields

$$\dot{\eta} = \tfrac{d}{dt}(x - x^*) = \dot{x},$$

since x^* is constant. Thus $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$. Now using Taylor's expansion we obtain

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2),$$

where $O(\eta^2)$ denotes quadratically small terms in η. Finally, note that $f(x^*) = 0$ since x^* is a fixed point. Hence

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2).$$

Now if $f'(x^*) \neq 0$, the $O(\eta^2)$ terms are negligible and we may write the approximation

$$\dot{\eta} \approx \eta f'(x^*).$$

This is a linear equation in η, and is called the ***linearization about*** x^*. It shows that *the perturbation* $\eta(t)$ *grows exponentially if* $f'(x^*) > 0$ *and decays if* $f'(x^*) < 0$. If $f'(x^*) = 0$, the $O(\eta^2)$ terms are not negligible and a nonlinear analysis is needed to determine stability, as discussed in Example 2.4.3 below.

The upshot is that the slope $f'(x^*)$ at the fixed point determines its stability. If you look back at the earlier examples, you'll see that the slope was always negative at a stable fixed point. The importance of the *sign* of $f'(x^*)$ was clear from our graphical approach; the new feature is that now we have a measure of *how* stable a fixed point is—that's determined by the *magnitude* of $f'(x^*)$. This magnitude plays the role of an exponential growth or decay rate. Its reciprocal $1/|f'(x^*)|$ is a ***characteristic time scale***; it determines the time required for $x(t)$ to vary significantly in the neighborhood of x^*.

EXAMPLE 2.4.1:

Using linear stability analysis, determine the stability of the fixed points for $\dot{x} = \sin x$.

Solution: The fixed points occur where $f(x) = \sin x = 0$. Thus $x^* = k\pi$, where k is an integer. Then

$$f'(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd}. \end{cases}$$

Hence x^* is unstable if k is even and stable if k is odd. This agrees with the results shown in Figure 2.1.1. ■

EXAMPLE 2.4.3:

What can be said about the stability of a fixed point when $f'(x^*) = 0$?

Solution: Nothing can be said in general. The stability is best determined on a case-by-case basis, using graphical methods. Consider the following examples:

(a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = 0$

Each of these systems has a fixed point $x^* = 0$ with $f'(x^*) = 0$. However the stability is different in each case. Figure 2.4.1 shows that (a) is stable and (b) is unstable. Case (c) is a hybrid case we'll call ***half-stable***, since the fixed point is attracting from the left and repelling from the right. We therefore indicate this type of fixed point by a half-filled circle. Case (d) is a whole line of fixed points; perturbations neither grow nor decay.

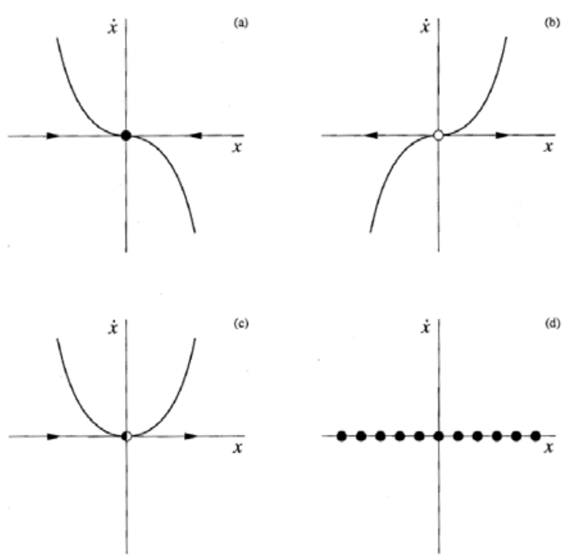

Figure 2.4.1