Bifurcations in 1st order nonlinear systems

(from Strogatz, Nonlinear dynamics and chaos, Perseus Books, Cambridge, Ma, 1994, Ch. 3)

As we've seen in Chapter 2, the dynamics of vector fields on the line is very limited: all solutions either settle down to equilibrium or head out to $\pm\infty$. Given the triviality of the dynamics, what's interesting about one-dimensional systems? Answer: *Dependence on parameters.* The qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called ***bifurcations***, and the parameter values at which they occur are called ***bifurcation points***.

3.1 Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are *created and destroyed.* As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

The prototypical example of a saddle-node bifurcation is given by the first-order system

$$\dot{x} = r + x^2 \qquad (1)$$

where r is a parameter, which may be positive, negative, or zero. When r is negative, there are two fixed points, one stable and one unstable (Figure 3.1.1a).

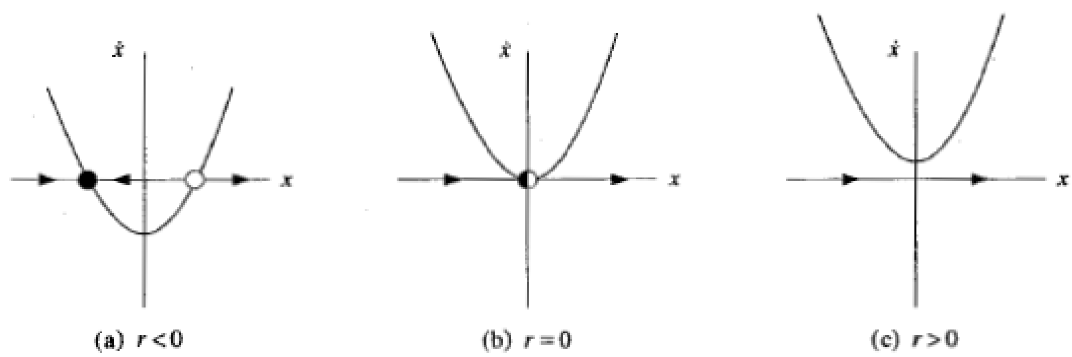


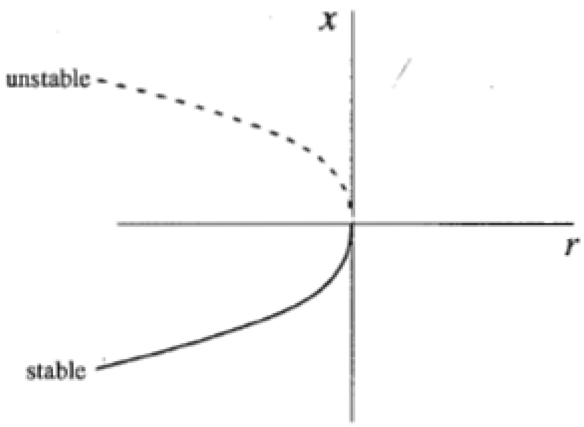
Figure 3.1.1

As r approaches 0 from below, the parabola moves up and the two fixed points move toward each other. When $r = 0$, the fixed points coalesce into a half-stable fixed point at $x^* = 0$ (Figure 3.1.1b). This type of fixed point is extremely delicate—it vanishes as soon as $r > 0$, and now there are no fixed points at all (Figure 3.1.1c).

In this example, we say that a *bifurcation* occurred at $r = 0$, since the vector fields for $r < 0$ and $r > 0$ are qualitatively different.

Graphical convention

There are several ways to depict a saddle-node bifurcations. The most common is reported in the figure below. To distinguish between stable and unstable fixed points, a ***solid line*** is used for stable points and a ***broken line*** for unstable ones.



This picture is called the ***bifurcation diagram*** for the saddle-node bifurcation.

Terminology

Bifurcation theory is rich in conflicting terminology and, frequently, different terms are used for the same thing . For example, the saddle-node bifurcation is sometimes called a *fold bifurcation* (because the curve in Figure 3.1.4 has a fold in it) or a *turning-point bifurcation* (because the point $(x, r) = (0,0)$ is a "turning point.") Admittedly, the term *saddle-node* doesn't make much sense for vector fields on the line. The name derives from a completely analogous bifurcation seen in a higher-dimensional context, such as vector fields on the plane, where fixed points known as saddles and nodes can collide and annihilate (see Section 8.1).

3.2 Transcritical Bifurcation

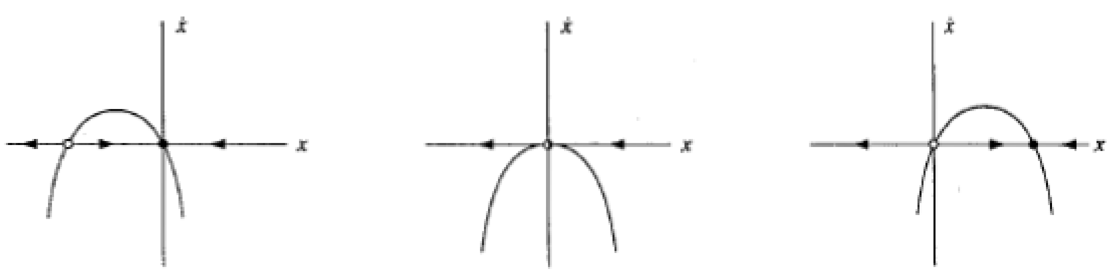
There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, regardless of the value of the growth rate. However, such a fixed point may *change its stability* as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2 . \qquad (1)$$

This looks like the logistic equation of Section 2.3, but now we allow x and r to be either positive or negative.

Figure 3.2.1 shows the vector field as r varies. Note that there is a fixed point at $x^* = 0$ for *all* values of r.



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Figure 3.2.1

For $r < 0$, there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when $r = 0$. Finally, when $r > 0$, the origin has become unstable, and $x^* = r$ is now stable. Some people say that an ***exchange of stabilities*** has taken place between the two fixed points.

Please note the important difference between the saddle-node and transcritical bifurcations: in the transcritical case, the two fixed points don't disappear after the bifurcation—instead they just switch their stability.

Figure 3.2.2 shows the bifurcation diagram for the transcritical bifurcation. As in Figure 3.1.4, the parameter r is regarded as the independent variable, and the fixed points $x^* = 0$ and $x^* = r$ are shown as dependent variables.

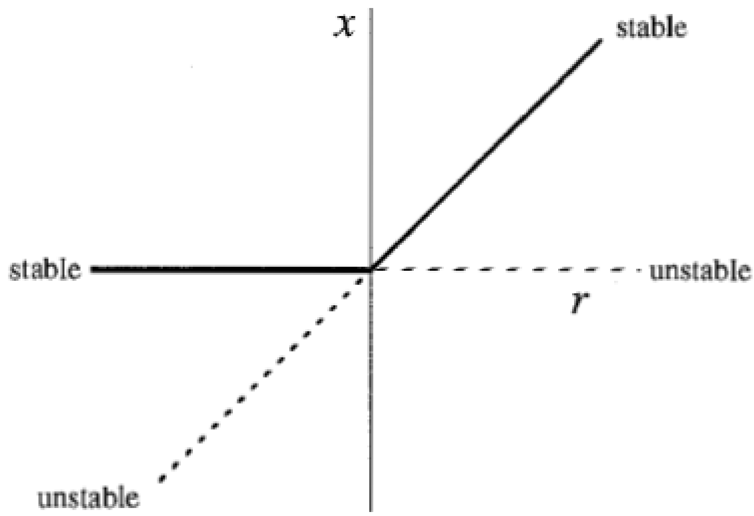


Figure 3.2.2

3.4 Pitchfork Bifurcation

We turn now to a third kind of bifurcation, the so-called pitchfork bifurcation. This bifurcation is common in physical problems that have a ***symmetry***. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs. In the buckling example of Figure 3.0.1, the beam is stable in the vertical position if the load is small. In this case there is a stable fixed point corresponding to zero deflection. But if the load exceeds the buckling threshold, the beam may buckle to *either* the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born.

There are two very different types of pitchfork bifurcation. The simpler type is called *supercritical*, and will be discussed first.

Supercritical Pitchfork Bifurcation

The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3 . \qquad (1)$$

Note that this equation is ***invariant*** under the change of variables $x \to -x$. That is, if we replace x by $-x$ and then cancel the resulting minus signs on both sides of the equation, we get (1) back again. This invariance is the mathematical expression of the left–right symmetry mentioned earlier. (More technically, one says that the vector field is *equivariant*, but we'll use the more familiar language.)

Figure 3.4.1 shows the vector field for different values of r.

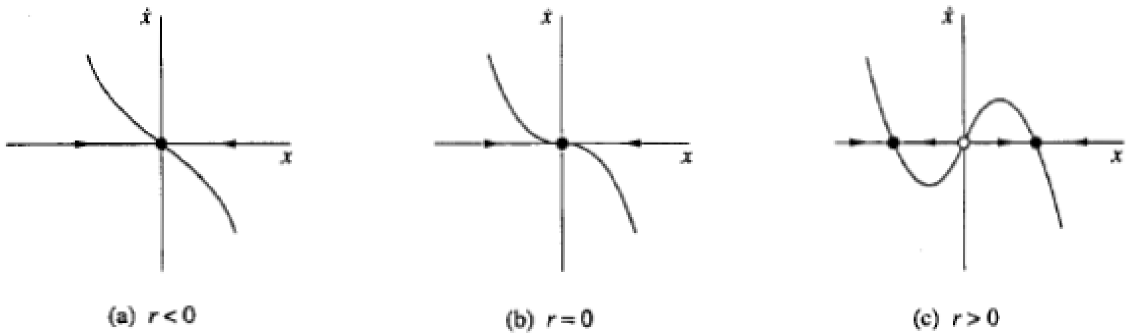


Figure 3.4.1

When $r < 0$, the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast—instead the decay is a much slower algebraic function of time (recall Exercise 2.4.9). This lethargic decay is called ***critical slowing down*** in the physics literature. Finally, when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$.

The reason for the term "pitchfork" becomes clear when we plot the bifurcation diagram (Figure 3.4.2). Actually, pitchfork trifurcation might be a better word!

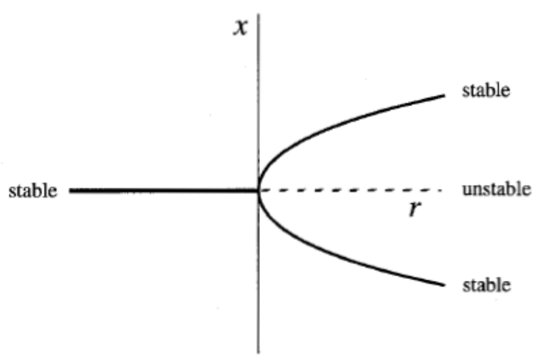


Figure 3.4.2

Subcritical Pitchfork Bifurcation

In the supercritical case $\dot{x} = rx - x^3$ discussed above, the cubic term is *stabilizing*: it acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were *destabilizing*, as in

$$\dot{x} = rx + x^3, \qquad (2)$$

then we'd have a ***subcritical*** pitchfork bifurcation. Figure 3.4.6 shows the bifurcation diagram.

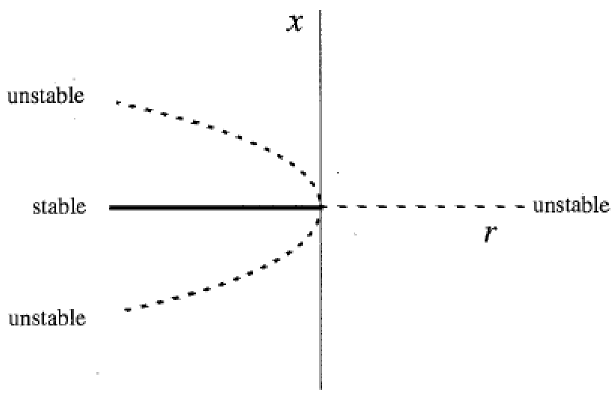


Figure 3.4.6

Compared to Figure 3.4.2, the pitchfork is inverted. The nonzero fixed points $x^* = \pm\sqrt{-r}$ are *unstable,* and exist only *below* the bifurcation ($r < 0$), which motivates the term "subcritical." More importantly, the origin is stable for $r < 0$ and unstable for $r > 0$, as in the supercritical case, but now the instability for $r > 0$ is not opposed by the cubic term—in fact the cubic term lends a helping hand in driving the trajectories out to infinity! This effect leads to *blow-up*: one can show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$ (Exercise 2.5.3).

In real physical systems, such an explosive instability is usually opposed by the stabilizing influence of higher-order terms. Assuming that the system is still symmetric under $x \to -x$, the first stabilizing term must be x^5. Thus the canonical example of a system with a subcritical pitchfork bifurcation is

$$\dot{x} = rx + x^3 - x^5. \qquad (3)$$

There's no loss in generality in assuming that the coefficients of x^3 and x^5 are 1 (Exercise 3.5.8).

The detailed analysis of (3) is left to you (Exercises 3.4.14 and 3.4.15). But we will summarize the main results here. Figure 3.4.7 shows the bifurcation diagram for (3).

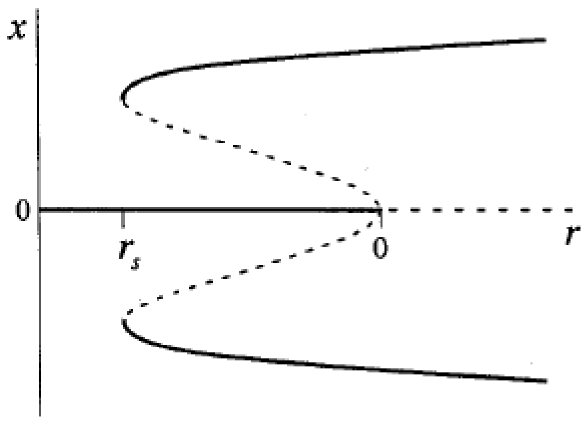


Figure 3.4.7

For small x, the picture looks just like Figure 3.4.6: the origin is locally stable for $r<0$, and two backward-bending branches of unstable fixed points bifurcate from the origin when $r=0$. The new feature, due to the x^5 term, is that the unstable branches turn around and become stable at $r=r_s$, where $r_s<0$. These stable ***large-amplitude*** branches exist for all $r>r_s$.

There are several things to note about Figure 3.4.7:

1. In the range $r_s<r<0$, two qualitatively different stable states coexist, namely the origin and the large-amplitude fixed points. The initial condition x_0 determines which fixed point is approached as $t\to\infty$. One consequence is that the origin is stable to small perturbations, but not to large ones—in this sense the origin is *locally* stable, but not *globally* stable.
2. The existence of different stable states allows for the possibility of ***jumps*** and ***hysteresis*** as r is varied. Suppose we start the system in the state $x^*=0$, and then slowly increase the parameter r (indicated by an arrow along the r-axis of Figure 3.4.8).

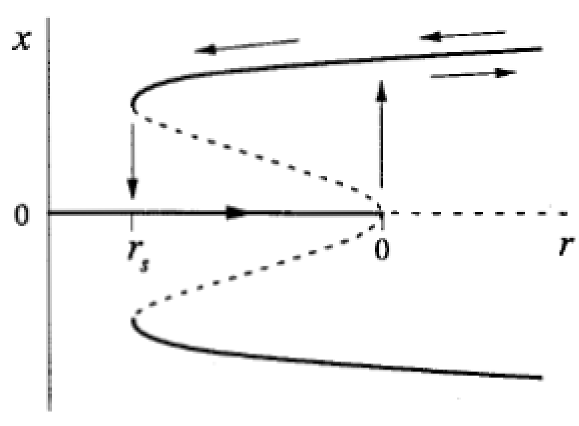


Figure 3.4.8

Then the state remains at the origin until $r = 0$, when the origin loses stability. Now the slightest nudge will cause the state to *jump* to one of the large-amplitude branches. With further increases of r, the state moves out along the large-amplitude branch. If r is now decreased, the state remains on the large-amplitude branch, even when r is decreased below 0! We have to lower r even further (down past r_s) to get the state to jump back to the origin. This lack of reversibility as a parameter is varied is called *hysteresis*.

3. The bifurcation at r_s is a saddle-node bifurcation, in which stable and unstable fixed points are born "out the clear blue sky" as r is increased (see Section 3.1).