1 The Chua's oscillator

The Chua's oscillator is shown in Fig. 1, together with a typical Chua's diode characteristic. Note that for $R_0 = 0$ we obtain the (classical) Chua's circuit.

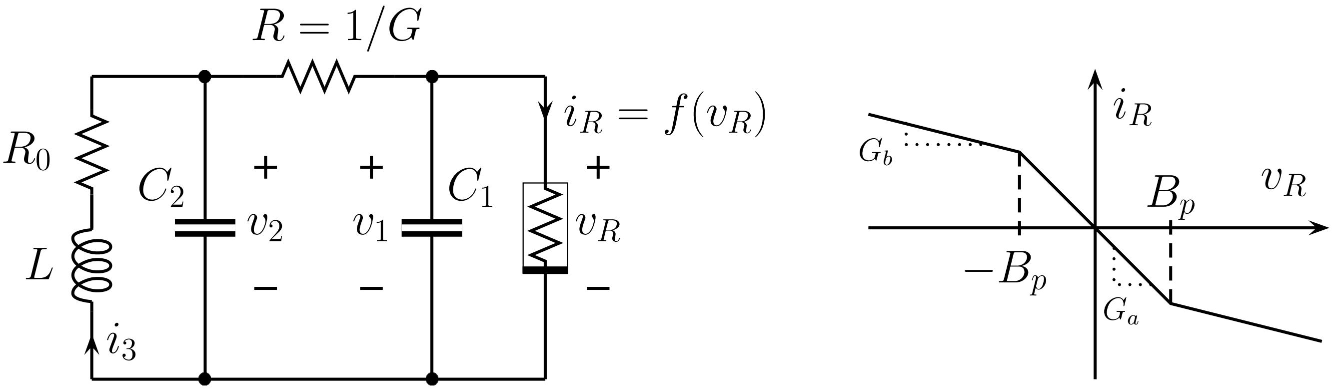


Figure 1: Chua's oscillator and Chua's diode

1.1 State equations

The circuit used to get the state equations of Chua's oscillator is shown in Fig. 2, where each capacitor has been substituted by a voltage source and the inductor by a current source.

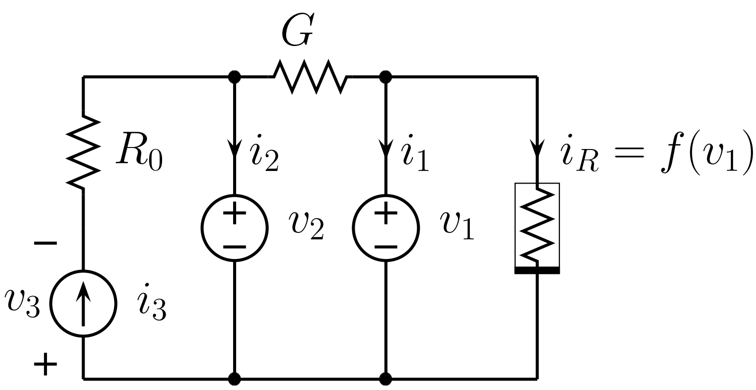


Figure 2: Circuit used to get the state equations of Chua's oscillator

The state equations are obtained by computing the currents i_1 and i_2 through the two voltage sources and the voltage v_3 across the current source and remembering that

$$i_1 = C_1 \frac{dv_1}{dt} \quad , \quad i_2 = C_2 \frac{dv_2}{dt} \quad , \quad v_3 = L \frac{di_3}{dt}$$

The state equations are:

$$\begin{cases} \dfrac{dv_1}{dt} & = & \dfrac{1}{C_1}\left[(v_2 - v_1)G - f(v_1)\right] \\ \dfrac{dv_2}{dt} & = & \dfrac{1}{C_2}\left[(v_1 - v_2)G + i_3\right] \\ \dfrac{di_3}{dt} & = & -\dfrac{1}{L}\left[v_2 + R_0 i_3\right] \end{cases}$$

with

$$f(v_1) = G_b v_1 + 0.5(G_a - G_b)[|v_1 + B_p| - |v_1 - B_p|]$$

1.2 Dimensionless state equations

Let us scale time, voltages, and currents by RC_2, B_p, and B_pG, respectively. Then let us assume as variables:

$$\tau = \frac{t}{RC_2} \ \ , \ x = \frac{v_1}{B_p} \ \ , \ y = \frac{v_2}{B_p} \ \ , \ z = \frac{i_3}{B_pG}$$

With these assumptions, we obtain the following dimensionless state equations:

$$\begin{cases} \dfrac{dx}{d\tau} & = & \alpha(-x + y - f(x)) \\ \dfrac{dy}{d\tau} & = & x - y + z \\ \dfrac{dz}{d\tau} & = & -\beta y - \gamma z \end{cases}$$

with

$$\alpha = \frac{C_2}{C_1} \ \ , \ \beta = R^2 C_2 / L \ \ , \ \gamma = R R_0 C_2 / L$$

and with

$$f(x) = m_1 x + 0.5(m_0 - m_1)[|x + 1| - |x - 1|] \ \ , \ m_0 = G_a R \ \ , \ m_1 = G_b R$$

Note that the case $\gamma = 0$ corresponds to the (classical) Chua's circuit.

1.3 Equilibrium points

The equilibrium points are obtained setting to zero the derivatives in the state equations. That means that the capacitors are substituted by open circuits and the inductor by a short circuit, leading to the circuit of Fig. 3. Graphically, the equilibrium points are obtained by intersecting the "load line" of slope $-1/(R + R_0)$ with the Chua's diode characteristic.

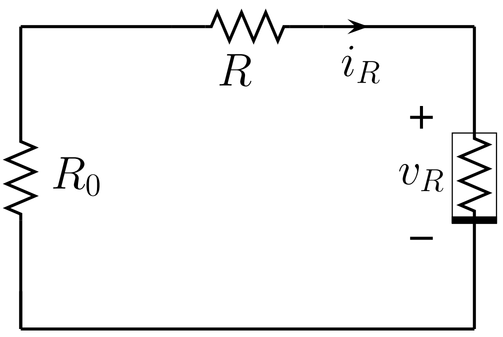


Figure 3: Circuit used to compute the equilibrium points

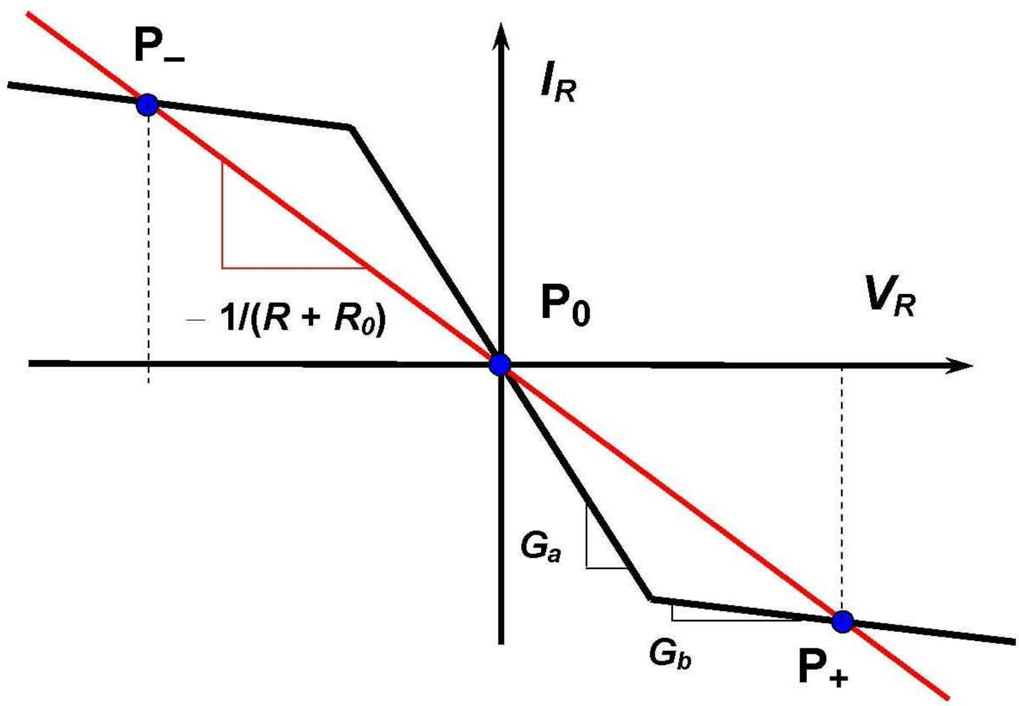


Figure 4: The equilibrium points are obtained by intersecting the "load line" of slope $-1/(R+R_0)$ with the Chua's diode characteristic.

2 Designing the Chua's diode

In this Section we will design a resistive one-port with a piecewise linear characteristic, to be used in the realization of the Chua's oscillator of Fig. 1. The operational amplifier is used as a basic element. Before giving any design formulas, let us briefly summarize the main properties of the ideal operational amplifier.

2.1 The ideal operational amplifier

The symbol of the ideal operational amplifier (op-amp) and its transfer characteristic are shown in Fig. 5. The equivalent circuits for linear and saturation regions are shown in Fig. 6.

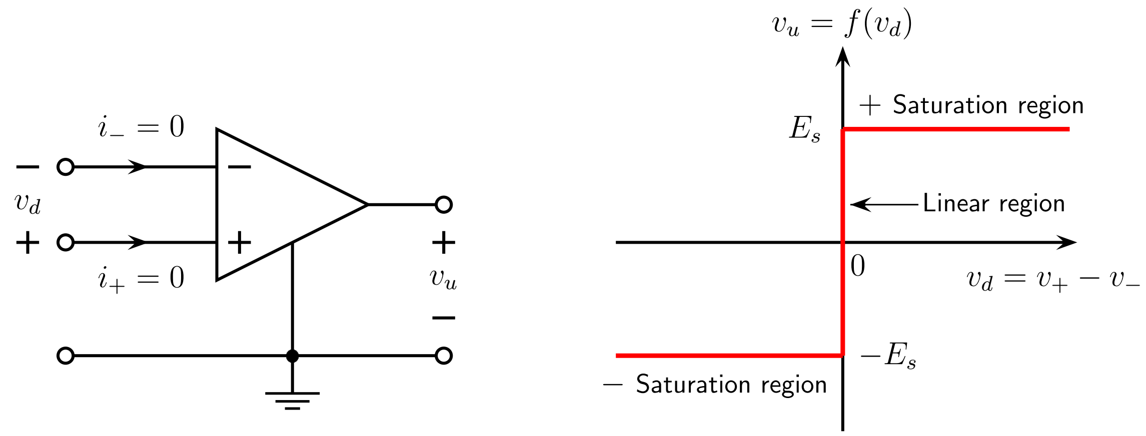


Figure 5: Symbol of the ideal operational amplifier and its transfer characteristic.

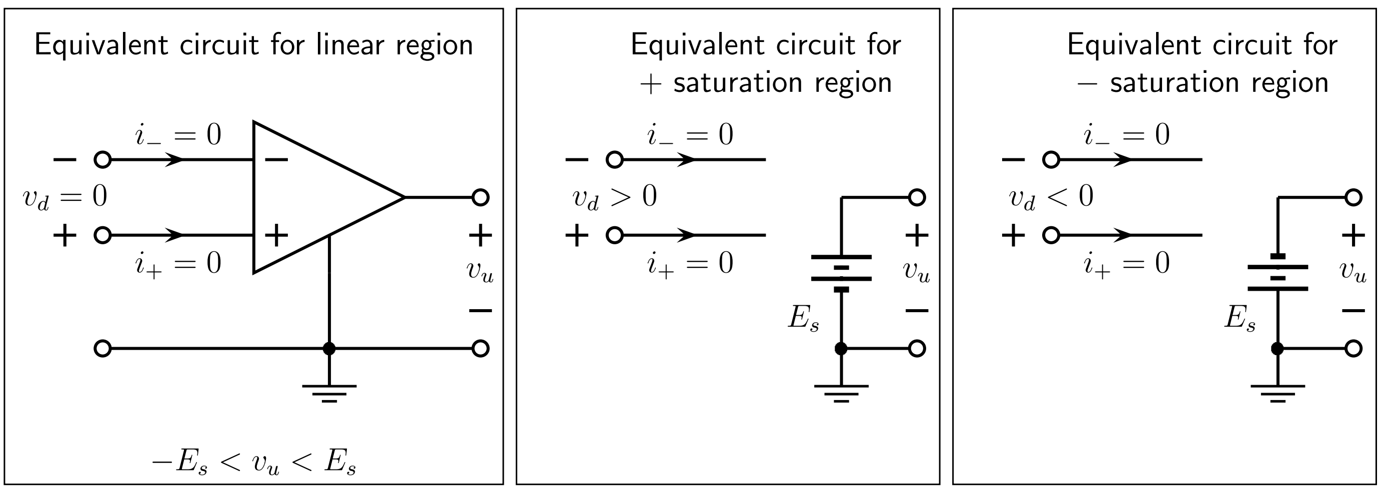


Figure 6: Equivalent circuits for linear and saturation regions.

2.2 A locally active piecewise resistor

The circuit used is shown in Fig. 7. Its input resistance R_e is given by the ratio v/i. To evaluate R_e, let us first observe that, due to the constraints imposed by the op amp, the voltage of the inverting input (with respect to ground) is v. Hence

$$i_3 = v/R_3$$

and

$$v_u = (R_2 + R_3)i_3 = v\frac{R_2 + R_3}{R_3}$$

Since

$$i = \frac{v - v_u}{R_1}$$

we have

$$i = -\frac{R_2}{R_1 R_3}v$$

and hence

$$R_e = v/i = -\frac{R_1 R_3}{R_2}$$

If we set $R_1 = R_2$ we get

$$R_e = -R_3$$

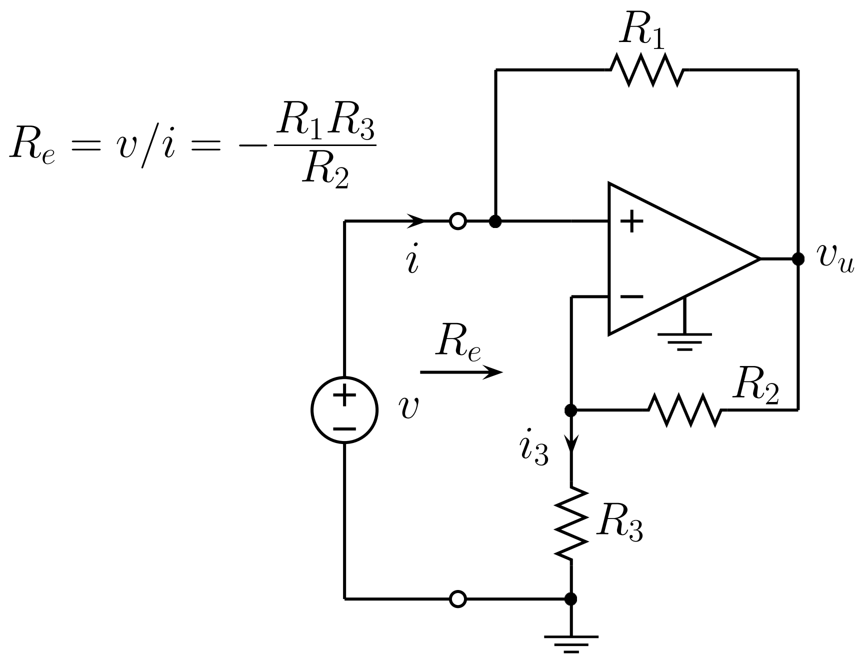


Figure 7: A circuit exhibiting a negative resistance $R_e = v/i = -\frac{R_1 R_3}{R_2}$

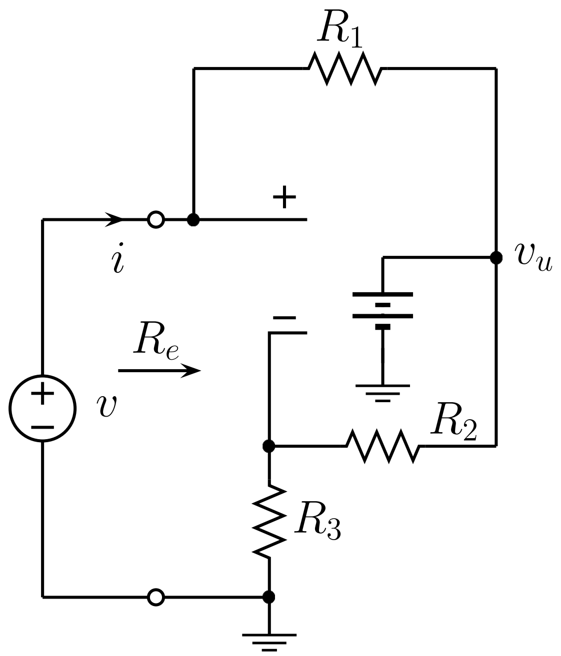


Figure 8: Equivalent circuit when the op amp works in its positive saturation region ($v > B_p = \dfrac{R_3}{R_2 + R_3} E_s$)

The above equations are valid as far as the op amp works in its linear region, that is as far as $|v_u| < E_s$, being E_s ($E_s > 0$) the saturation voltage of the op amp. Taking into account the expression of v_u, the op amp works in its linear region as long as

$$|v| < \frac{R_3}{R_2 + R_3} E_s$$

The breakpoint voltages of the $v \div i$ characteristic are symmetric with respect to the origin and the positive one, B_p, is given by

$$B_p = \frac{R_3}{R_2 + R_3} E_s$$

When $v > B_p$ the op amp enters its positive saturation region. The equivalent circuit is given in Fig. 8 and the expression of i becomes

$$i = \frac{v}{R_1} - \frac{E_s}{R_1}$$

A similar reasoning holds when $v < -B_p$: in this case the op amp enters its negative saturation region and now the expression of i becomes

$$i = \frac{v}{R_1} + \frac{E_s}{R_1}$$

The whole characteristic is shown in Fig. 9

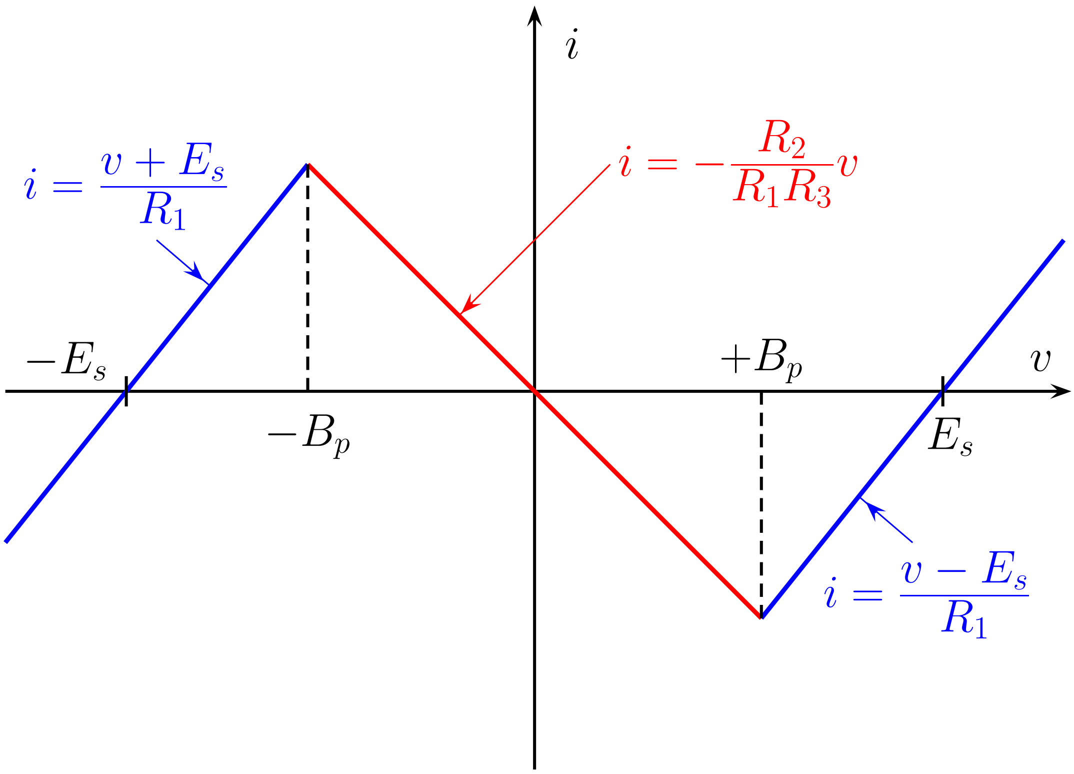


Figure 9: Locally-active piecewise resistor characteristic

2.3 Design formulas of the piecewise resistor

According to the results of the previous section and assuming that the saturation voltage E_s of the op amp is known, the inner slope G_a, the outer slope G_b, and the breakpoint voltage B_p (see Fig. 10) are given by

$$G_a = -\frac{R_2}{R_1 R_3}, \quad G_b = \frac{1}{R_1}, \quad B_p = \frac{1}{1 + \frac{R_2}{R_3}} E_s$$

Now let us suppose that the inner slope $G_a \ (G_a < 0)$, the outer slope G_b, and the value of the breakpoint voltage B_p are assigned. The design formulas can be obtained from the equations above. If we assume $R_1 = R_2$, from the first two equations we get

$$\begin{cases} R_1 = R_2 & = & \frac{1}{G_b} \\ R_3 & = & -\frac{1}{G_a} \quad (G_a < 0) \end{cases}$$

The third equation links the breakpoint voltage B_p to the saturation voltage E_s. Note that E_s is assumed known and fixed at a particular value, that depends on the internal structure of the op amp and on the used power supply voltages. It is not known *a priori* but it can be measured. As a consequence, it turns out that the value of B_p cannot be fixed independently from the ratio G_a/G_b

(even if R_2 is not chosen equal to R_1).

$$B_p = \frac{E_s}{1 + \dfrac{R_2}{R_3}} = \frac{E_s}{1 - \dfrac{G_a}{G_b}}$$

or, equivalently

$$E_s G_b = B_p (G_b - G_a)$$

Finally, it is worth noting to observe that the above equation also states the continuity of the characteristic at the breakpoint B_p.

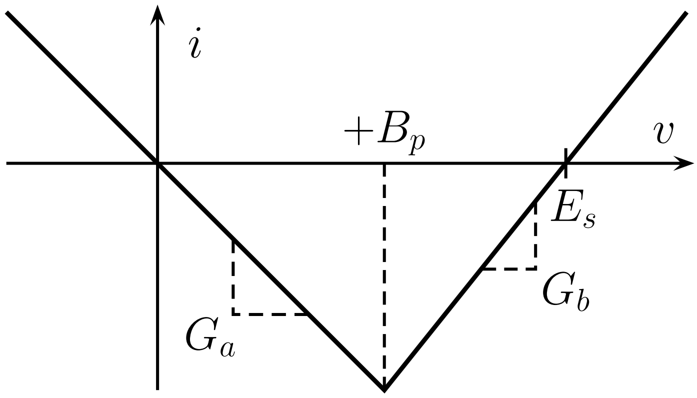


Figure 10: Diode design parameters

2.4 The Chua's diode

The Chua's diode can be obtained by the parallel connection of two piecewise nonlinear resistors of the kind described in the previous Subsection, as shown in Fig. 11. Since the voltage v across the two resistors D_1 and D_2 is the same, the resulting characteristic is obtained by summing the currents i_1 and i_2 for equal values of v, as shown in Fig. 12

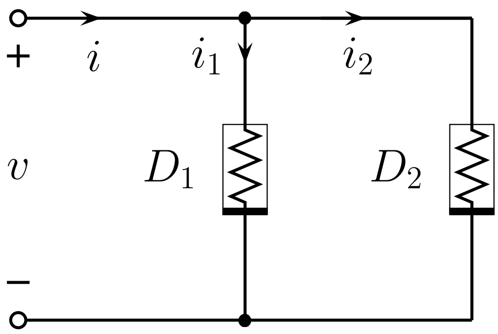


Figure 11: Two piecewise linear resistor connected in parallel

The complete circuit is shown in Fig. 13, along with the analysis and design equations for each diode.

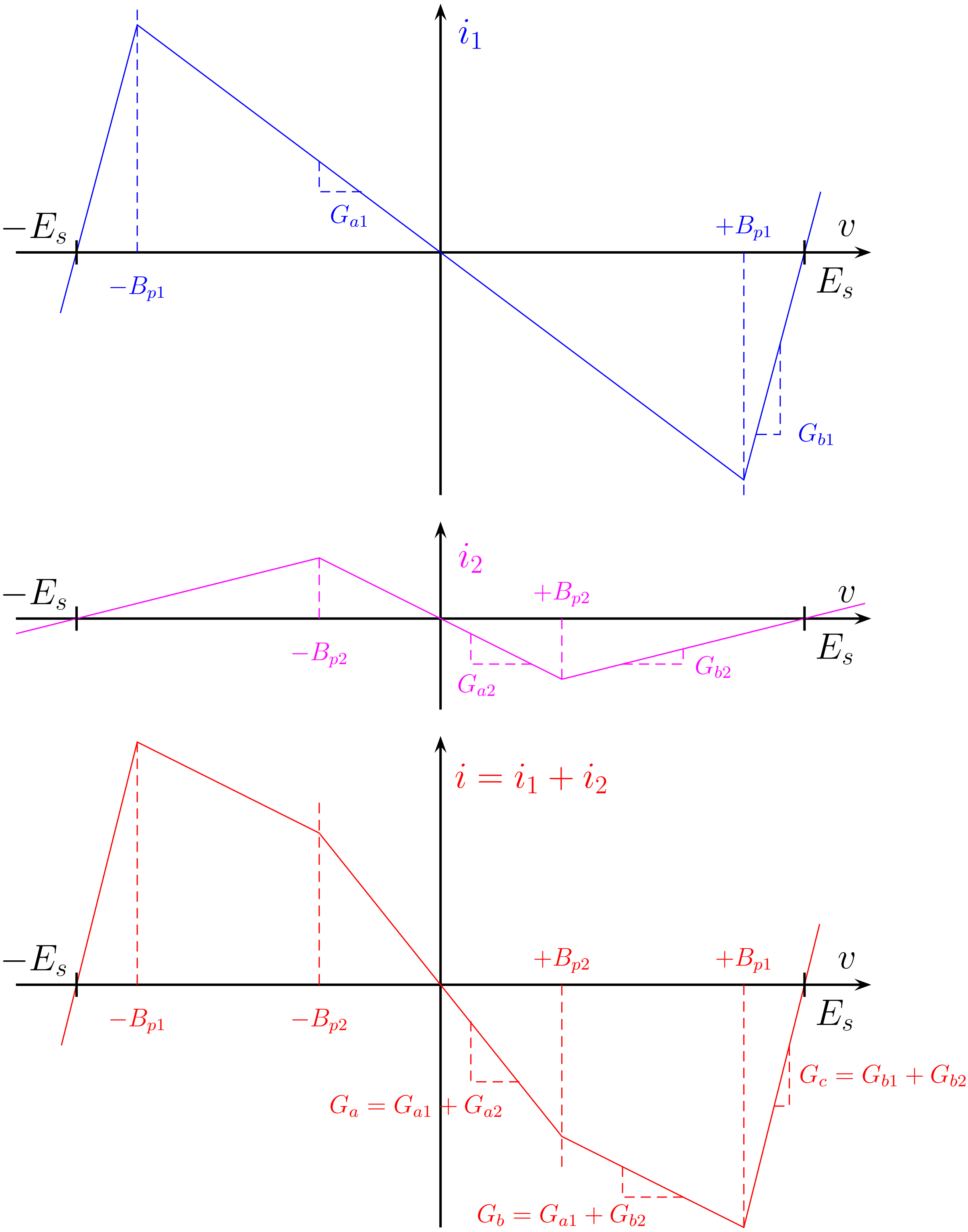
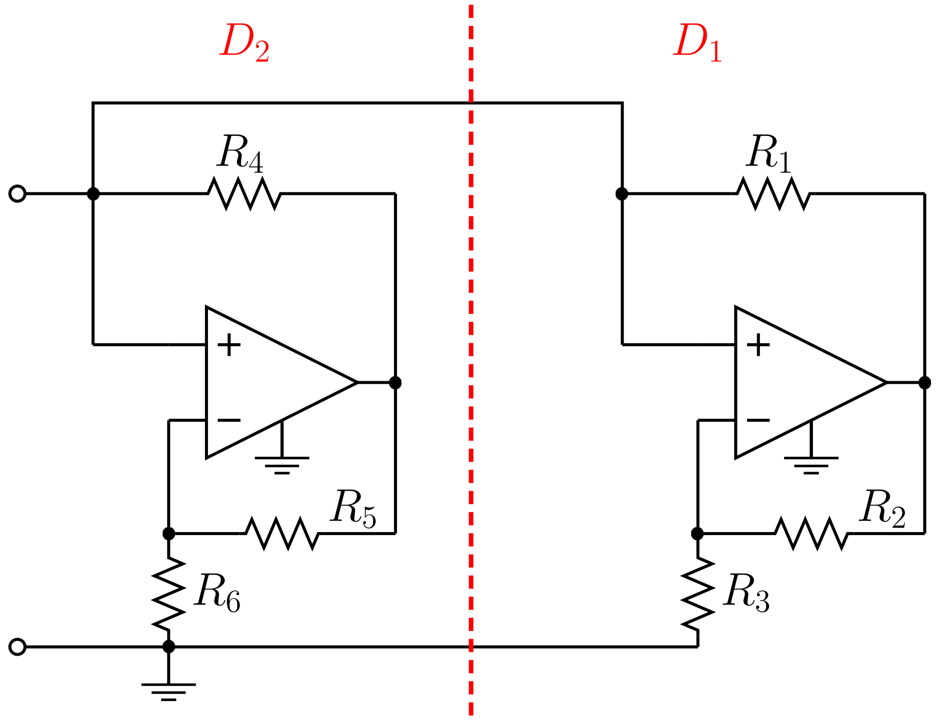


Figure 12: Chua's diode characteristic via parallel connection of two piecewise linear resistors



ANALYSIS EQUATIONS:

$$G_{a2} = -\frac{1}{R_6}, \quad (R_4 = R_5)$$

$$G_{b2} = \frac{1}{R_4}$$

$$B_{p2} = \frac{E_s}{1 + \dfrac{R_5}{R_6}}$$

$$G_{a1} = -\frac{1}{R_3}, \quad (R_1 = R_2)$$

$$G_{b1} = \frac{1}{R_1}$$

$$B_{p1} = \frac{E_s}{1 + \dfrac{R_3}{R_2}}$$

DESIGN EQUATIONS:

$$R_4 = R_5 = \frac{1}{G_{b2}}$$

$$R_6 = -\frac{1}{G_{a2}}$$

$$R_1 = R_2 = \frac{1}{G_{b1}}$$

$$R_3 = -\frac{1}{G_{a1}}$$

with the constrain:

$$B_{p2} = \frac{E_s}{1 - \dfrac{G_{a2}}{G_{b2}}}$$

$$B_{p1} = \frac{E_s}{1 - \dfrac{G_{a1}}{G_{b1}}}$$

Figure 13: Chua's diode and analysis and design equations of PWL resistors D_1 and D_2

2.5 Evaluation of the design parameters

Referring to Fig. 12, the shape of the desired characteristic is determined by G_a, G_b, B_{p2}, and B_{p1}. The value of B_{p1} only influences the extension of the negative-slope region and, hence, it is less relevant for the final design. Furthermore, the saturation voltage E_s is supposed to be known and equal for both the op amps.

To design the Chua's diode, we must first compute the values of the slopes G_{a1} , G_{b1} , G_{a2} , and G_{b2} of the two diodes $D1$ and $D2$. First, from Fig. 12 we obtain the following two equations

$$\begin{cases} G_{a1} + G_{a2} & = & G_a \\ G_{a1} + G_{b2} & = & G_b \end{cases}$$

Furthermore, the following two constraints (see Fig. 13) hold

$$\begin{cases} (E_s - B_{p2})G_{b2} + B_{p2}G_{a2} & = & 0 \\ (E_s - B_{p1})G_{b1} + B_{p1}G_{a1} & = & 0 \end{cases}$$

Altogether, we obtain the following system of four equations in the four unknown G_{a1} , G_{b1} , G_{a2} , and G_{b2}

$$\begin{cases} G_{a1} + G_{a2} & = & G_a \\ G_{a1} + G_{b2} & = & G_b \\ (E_s - B_{p2})G_{b2} + B_{p2}G_{a2} & = & 0 \\ (E_s - B_{p1})G_{b1} + B_{p1}G_{a1} & = & 0 \end{cases}$$

Since G_{b1} does not appear in the first three equations of the system, they can be solved independently from the fourth equation. We obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & B_{p2} & E_s - B_{p2} \end{bmatrix} \begin{bmatrix} G_{a1} \\ G_{a2} \\ G_{b2} \end{bmatrix} = \begin{bmatrix} G_a \\ G_b \\ 0 \end{bmatrix}$$

Solving this system we obtain

$$\begin{aligned} G_{a1} &= \frac{G_a B_{p2} + G_b(E_s - B_{p2})}{E_s} \\ G_{a2} &= \frac{(E_s - B_{p2})(G_a - G_b)}{E_s} \\ G_{b2} &= \frac{B_{p2}(G_b - G_a)}{E_s} \end{aligned}$$

and, from the fourth equation

$$G_{b1} = -\frac{B_{p1}}{E_s - B_{p1}} G_{a1}$$

Using the design equations listed in Fig. 13 it is now possible to compute the element values of the Chua's diode, as explained in the following Subsection.

2.6 Design procedure

We are now able to suggest a design procedure for the Chua's diode. Once again we would like to stress that the value of B_{p1} is not critical and it only should be large enough to ensure that the dynamics of the attractor remains within the negative-resistance region of the whole characteristic. Then the design procedure can follow these five steps

1. Choose
$$R_1 = R_2 \text{ and } R_4 = R_5$$
2. Evaluate
$$R_3 = -\frac{1}{G_{a1}} = \frac{E_s}{G_b(B_{p2} - E_s) - G_a B_{p2}}$$
3. Evaluate
$$R_6 = -\frac{1}{G_{a2}} = \frac{E_s}{(E_s - B_{p2})(G_b - G_a)}$$
4. Evaluate
$$R_4 = \frac{1}{G_{b2}} = \frac{E_s}{B_{p2}(G_b - G_a)}$$
Note that this value is generally high, since the difference $G_b - G_a$ is typically small, and hence the resistors R_4 and R_5 do not load significantly the op amp.
5. Evaluate
$$R_1 = \frac{1}{G_{b1}} = -\frac{E_s - B_{p1}}{B_{p1}} \frac{1}{G_{a1}} = (1 - \frac{E_s}{B_{p1}}) \frac{1}{G_{a1}}$$
If the obtained value of R_1 (and R_2) is too small and loads the op amp, we can assign smaller values to B_{p1} until a trade off is reached between the length of the negative-resistance region and the size of R_1. Generally, to use off-the-shelf components, one assigns a suitable value to R_1 and then evaluates B_{p1}
$$B_{p1} = E_s/(1 - R_1 G_{a1})$$
If this value is too small, then a new (smaller) value of R_1 is used.

2.7 An example

Assume that the op amps are powered by two 9 V batteries and that the saturation voltage equals 8.3 V. The required slopes in the negative-resistance region – capable to assure interesting dynamic behaviors – are chosen to be $G_a = -0.756\,\text{mS}$ and $G_b = -0.409\,\text{mS}$. The inner-breakpoint value is set to $B_{p2} = 1.08\,\text{V}$. According to the outlined procedure we get

$$\begin{aligned} R_3 &= 2.2\,\text{k}\Omega \\ R_6 &= 3.31\,\text{k}\Omega \cong 3.3\,\text{k}\Omega \\ R_4 = R_5 &= 22.15\,\text{k}\Omega \cong 22\,\text{k}\Omega \end{aligned}$$

After some trials, we chose
$$R_1 = R_2 = 220\,\Omega$$

The value of the breakpoint B_{p1} turns out to be high enough, compared to a saturation voltage of 8.3 V

$$B_{p1} = 7.55\,\mathrm{V}$$

3 Designing the Chua's oscillator

To complete the realization of a Chua's oscillator, we have still to chose the values of the two capacitors $C1$, $C2$, and of the inductor L. The resistor R_0 can be assumed equal to the measured leakage resistor which models the loss of the real inductor L, whereas the resistor R is used as bifurcation parameter and hence it is varied in a quite wide range of values. Furthermore, we have to fix the characteristic of the Chua's diode, that is the values of G_a, G_b, B_{p2}, and (less critical) of B_{p1}.

All these values should be fixed in such a way that an interesting dynamic behavior is obtained as R is varied and, possibly, the circuit elements have suitable values, so that off-the-shelf components can be used.

In the first reported study [1] of the classical Chua's circuit (i.e. a Chua's oscillator with $R_0 = \gamma = 0$), a strange attractor is obtained by simulating a system of differential equations using the following set of (scaled) component values:

$$C_1 = 1/9\,, \quad C_2 = 1\,, \quad L = 1/7\,, \quad R = 10/7\,,$$

$$B_{p2} = 1\,, \quad B_{p1} \to \infty\,, \quad G_a\,\text{(inner region slope)} = -0.8\,, \quad G_b\,\text{(outer region slope)} = -0.5$$

These values correspond to the following set of dimensionless parameters:

$$\alpha = \frac{C_2}{C_1} = 9\,,\ \beta = R^2 C_2/L = 100/7 \cong 14.286\,,\ \gamma = R R_0 C_2/L = 0\,,\ m_0 = -5/7\,,\ m_1 = -8/7$$

Even if the Chua's oscillator is considered (with small values of γ, i.e. of R_0), extended simulations show that interesting dynamical behaviors are still observed if similar dimensionless parameter values are used. To use standard component values, we arbitrarily choose $C_2 = 100\,\mathrm{nF}$ and $C_1 = 10\,\mathrm{nF}$, that corresponds to $\alpha = 10$. Furthermore, we choose $L = 18\,\mathrm{mH}$ and the variable resistor R in the range of $2\,\mathrm{k\Omega}$, that approximately corresponds to the value of β used in [1].

Finally, we use slightly different values for G_a, G_b and B_{p2} in order to get standard resistor values for the Chua's diode. The value of B_{p1} is less critical: it is not fixed "*a priori*" and is accepted as long as it is sufficiently high, as explained in the previous Section.

The choice of these values is in part related to the op amp saturation voltage and is generally done in an heuristic way. For example, in a real circuit where two 9 V batteries are used to power the op amps, the (measured) saturation voltage equals 8.3 V. In this case, after a few trials, we get suitable component values (see Subsection 2.7) assuming $G_a = -0.756\,\mathrm{mS}$, $G_b = -0.409\,\mathrm{mS}$ and setting the inner-breakpoint B_{p2} to 1.08 V.

The complete list of the circuit components is given in Tab. 1

Table 1: Component List

Element	Value
R_1	$220\,\Omega$
R_2	$220\,\Omega$
R_3	$2.2\,\mathrm{k\Omega}$
R_4	$22\,\mathrm{k\Omega}$
R_5	$22\,\mathrm{k\Omega}$
R_6	$3.3\,\mathrm{k\Omega}$
C_1	$10\,\mathrm{nF}$
R	$2.0\,\mathrm{k\Omega}$
C_2	$100\,\mathrm{nF}$
L	$18\,\mathrm{mH}$
R_0 (measured)	$14\,\Omega$

References

[1] T. Matsumoto, "A chaotic attractor from Chua's circuit", *IEEE Trans. Circuits Syst.*, vol. CAS-31, pp. 1055–1058, 1984.